Measuring tidal variations in ‘g’ using Trinity’s pendulum clock

by

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Fourth-year undergraduate project in Group C

2008/2009

I hereby declare that, except where specifically indicated, the work submitted herein is my own original work.

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Abstract

In this project a system has been developed to monitor the performance of the 100-year-old mechanical pendulum clock in Trinity College Great Court. A previous project had developed techniques for measuring the running of the clock. This project has built on that work to produce a finished reliable system; much work has been done to improve the accuracy and reliability of the software. The system measures the clock’s drift (the difference between clock time and true time) and the pendulum’s amplitude, as well as the air temperature, humidity and pressure in the clock enclosure.

This instrumentation was used to calibrate a set of adjustment weights for the clock; these, together with the ability to instantaneously measure the error in the rate of the clock, make keeping the clock on time much easier.

Surprisingly interesting physical effects arise from a simple pendulum. First there is the fairly standard adjustment for non-linearity, which means the period increases as the amplitude of the swing increases. The amplitude in turn depends on the energy dissipated in drag, and so varies with air density. The pendulum is temperature-compensated, which tries to ensure its length is unchanged by thermal expansion; but there are time delays involved which mean transient temperature changes cause the clock to gain or loose time. The gravitational pull of the Moon and the Sun will also have an effect. In the second part of the project these effects have been analysed, producing a fairly comprehensive overview of the clock’s behaviour in response to variations in pendulum amplitude, mass distribution and length; air temperature and density; and draughts and tidal variations in gravity.

These theoretically predicted behaviours have been checked against the real data from the clock. The predictions of the response to changes in pendulum amplitude, mass distribution, air temperature and air density were shown to be quite accurate, while the effect of draughts and changes in pendulum length could not be confirmed due to insufficient data.

The predicted tidal variations turned out to be much smaller than preliminary estimates (causing at most a variation in rate of $\pm 10\,\mathrm{ms/day}$), too small to yet have been measured affecting the clock. However, the predictions are consistent with published measurements of gravity taken in Oklahoma, USA, which validates the theory. Spectral analysis theory was used to predict how much data will necessary to distinguish this small effect from the background noise: perhaps of the order of 100 days. Unfortunately, to date only about 30 days of continuous recorded data are available, so measuring the effect of the tidal variation on the clock will have to wait.

A website has been constructed to allow easy access to the data logged from the clock. Using it, one can browse through, plot and download any part of the data. It can be used by the clock keeper (for checking the status of the clock remotely), by those studying the clock (to easily find interesting events in the data), or by anyone else in the world (making the Trinity Clock a public 'pendulum laboratory').

There remain events in the recorded data which cannot be explained by the theoretic relationships developed here. Some examples of these are presented, and possible causes discussed.

Finally, possible further work is suggested: in particular there is plenty of scope for adding new sensors, which would provide additional data that might explain the currently unexplainable effects.

Contents

List of Figures

Nomenclature

A Pendulum amplitude (radians).

D Drift: clock time minus true time.

G Going: the rate of the clock, i.e. seconds gained per second.
Usually given in units of milliseconds per day, with 100 ms/day = 1.16 ppm (parts per million).

g Acceleration due to gravity.

GPS Global Positioning System.

IR Infrared: signal from the optical sensor on the pendulum.

L Pendulum length (≈ 2.4 m).

p Air pressure (hPa). 1 hPa = 1 mbar.

PPS Pulse Per Second: timing signal from GPS receiver.

RH Relative Humidity (%)

T Pendulum period [or occasionally temperature].

T_0 Nominal pendulum period = 3 seconds.

T_s Period of a simple pendulum (for small oscillations) $= 2\pi\sqrt{L/g}$.

Acknowledgements

Thanks to the Mechanics Laboratory technicians, in particular to Gary Bailey for constructing, revising and cheerfully re-revising the interface electronics, to Gareth Ryder for making a very nice infrared sensor fitting and set of calibrated brass weights, and to David Miller; to GianCarlo Pacitti and the other students who followed him, for their work developing techniques for measuring the running of the clock, which this project builds upon; also, of course, many thanks to my supervisor, Dr Hugh Hunt, for his support and many interesting discussions.

Richard Lupton
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1 Introduction

There have been a succession of clocks in Trinity Great Court since 1610 [1], with the latest one approaching its 100th anniversary next year. This current mechanism is a beautiful piece of engineering; it can still keep time quite accurately, to within 1 second per month, powered by falling weights and regulated by a pendulum. It is unusual in that it strikes the hours twice (according to legend, once for Trinity and once for St John's); it makes an appearance in William Wordsworth's poetry[1], and it is famous as the timekeeper for the "Great Court Run."

Although the clock is capable of keeping time quite accurately, it does needs adjustment. However, knowing instantaneously what adjustment is required is a bit of a challenge: even a relatively large error in the rate of the clock of, say, 1 second per day amounts to only an extra 35 μs in every swing of the pendulum. Since this is rather small, traditionally the clock keeper would wait until the end of the week, when the error will have built up enough to be measurable by comparing the clock with a radio-controlled wall clock. Then the clock can be adjusted by adding or removing small weights on the pendulum.

It would be preferable if the rate of the clock could be measured, and corrected, without waiting for the error to accumulate to noticeable levels, and this is the problem which past work on this project has aimed to solve. A previous 4th year project developed a system to measure the "going" (the rate of the clock, i.e. seconds gained per second) from both the sound of the ticks and an optical sensor on the pendulum [2]. This project builds on that work, aiming to produce a finished monitoring system which is accurate enough to measure all the variations in the clock's running, and reliable enough to run continuously for months on end.

The ability to instantaneously measure the "going" of the clock will make keeping the clock on time much easier. Using this ability, we can make a set of calibrated weights to

[1] William Wordsworth, *Residence at Cambridge,* The Prelude (1850), Book Third.

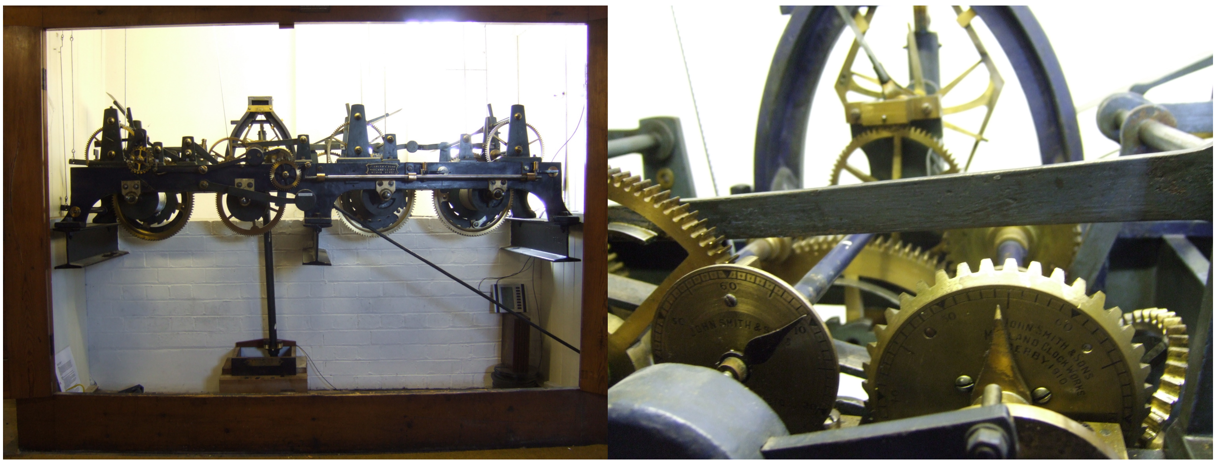


FIGURE 1.1: The Clock

replace the assorted washers currently in use; with these two developments, if the clock is running slow or fast the exact correction can be applied immediately.

But there is a whole other side to the project: when one starts to look at the resulting data, there are a multitude of interesting physical effects happening. Changes in going can be attributed to variations in pendulum amplitude, air density, temperature, air currents... even the gravitational pull of the Moon, as in the title of the project. To borrow a phrase from Nelson and Olsson [3], we can explore 'rich physics from a simple system.' The aim is to make a fairly comprehensive analysis of the types of behaviour arising from these effects. We can then draw on the recorded data from the clock to demonstrate these behaviours in the real world.

Also, a means of organising the fairly large quantities (one reading per tick) of data is needed. The aim here is to produce a website where the data can be browsed online, and selected parts downloaded. This will be useful not only to those studying or looking after the clock, who will be able to check its status from anywhere, but also will open up the rich physics of the pendulum to anyone in the world, turning the Trinity Clock into a public 'pendulum laboratory.'

In summary, the aims of the project are to:

1. Finalise the instrumentation and techniques for measuring the clock, resulting in a reliable set-up which can monitor the clock indefinitely;

2. Make the data freely available on a website for investigation;

3. Aid the clock keeper's work by, for example, producing calibrated regulation weights;

4. Model the behaviour of the clock in response to environmental variations, and compare against the real-world data.

2 Measurement Apparatus

2.1 Reference time source

Throughout their history there have always been methods of checking the running of clocks, ranging from sundials to modern atomic oscillators. For this project, the accuracy of the measuring system must be quite high: for example, a typical situation would be measuring the clock over the period of [ideally] uninterrupted running between the start and end of Daylight Savings Time in March and October. Over these 7 months, we might expect the clock to stay within 10 seconds of the true time, which represents an accuracy of 0.5 parts per million (ppm). The reference time source must, of course, be better than this.

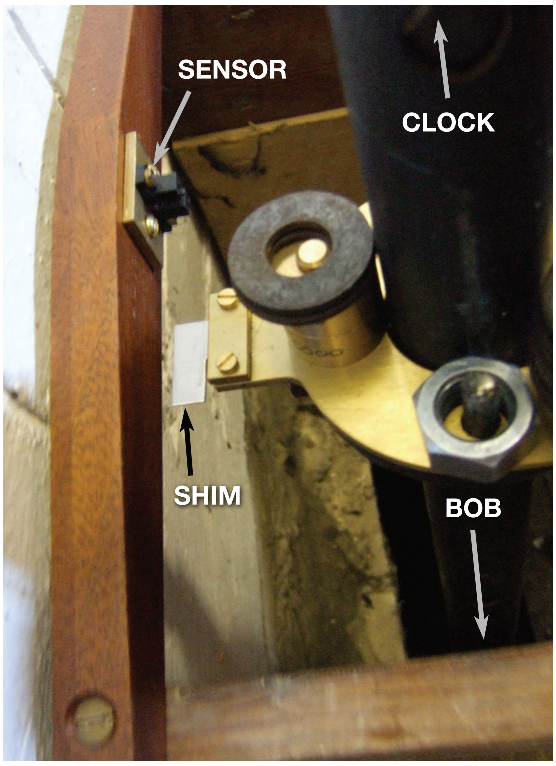


FIGURE 2.1: Photo of optical sensor and shim on pendulum.

A simple quartz oscillator was tested, but was not nearly good enough (it ran ~5 seconds per day, or 60 ppm, too fast). A better time reference is the Global Positioning System (GPS). The GPS satellites have very accurate clocks, required for position fixing, and the receiver used[1] gives an pulse output with an accuracy of ±1 microsecond [4]. Although ±1 μs is only ±1 ppm, unlike the quartz oscillator it is within 1 ppm of true time *every second*. Over 7 months the accuracy will be 10^{-7} ppm! At least 3 satellites must be visible before the time is known accurately, so the receiver is placed on the roof or in a window with a good view of the sky. As well as the pulse per second (PPS) output for timing, the serial connection to the GPS is used for configuration of the receiver and monitoring of the number of satellites visible.

2.2 Sensor

The next part of the apparatus is the sensor. Acoustic, optical, magnetic and inductive sensors have all been used to measure clocks [2, 5], but here we have continued to use the infrared (IR) optical sensor used previously.

The sensor is arranged such that a shim on the pendulum breaks the beam twice per period (see Figure 2.1), giving a signal with 4 edges per period; a typical recording of the input signals is shown in the upper part of Figure 2.2. To calculate the amplitude some

[1] a Garmin GPS 18x LVC

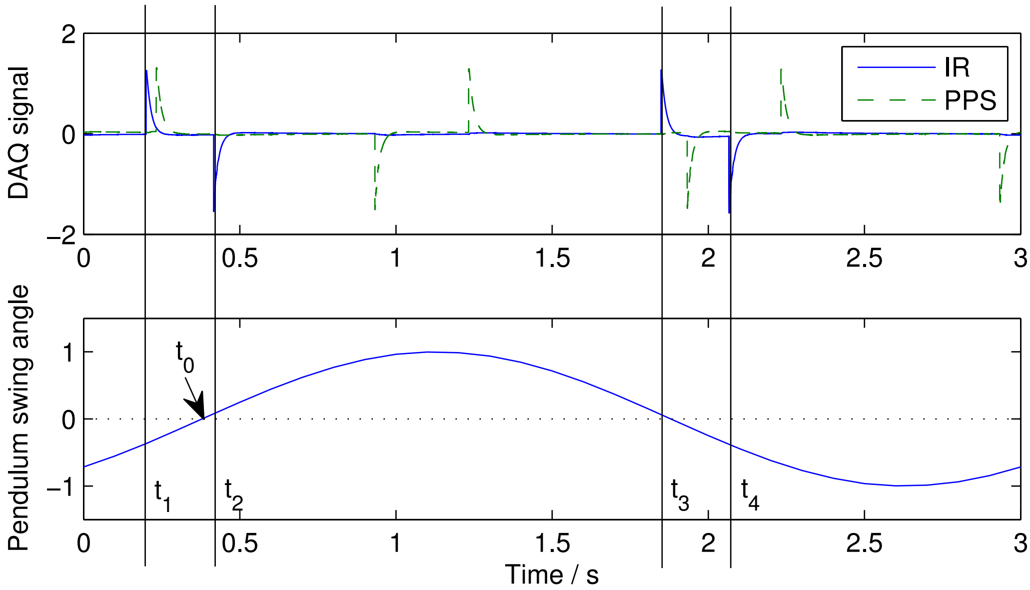


FIGURE 2.2: Typical input signals from IR sensor and GPS (high-pass filtered, above), and corresponding pendulum position (below).

asymmetry is needed in the position of the edges. As the sensor is located at the centre of the pendulum's swing, the shim must be mounted off-centre — the resulting IR edges are related to the pendulum position is shown in the lower part of Figure 2.2.

2.2.1 Optimisation of sensor / shim location

The size and location of the shim determines the points in the swing at which the IR sensor is triggered, and these times are used to calculate the amplitude of the pendulum (using Equations (2.1) & (2.2) below).

There will be some uncertainty in the trigger positions (due, for example, to variable ambient light levels), and the sensitivity of the amplitude calculation to these variations depends on the position of the trigger points in the swing, so there will be an optimum shim design. Equations (2.1) & (2.2) are not readily differentiable, so a Monte-Carlo analysis was used to find the optimum design. The resulting contours of error sensitivity are shown in Figure 2.3.

This shows that maximum accuracy is achieved for wide and highly asymmetric shims. The best design would be one which blocks the beam for almost the entire half-swing, allowing a brief pulse through at the top end of the swing. However, the amplitude of the pendulum changes both during day-to-day running and, more significantly, whenever the clock is stopped, intentionally (for example, for an hour at the end of Daylight Saving Time in October) or not. We would like to be able to measure the characteristics of the decaying oscillations, so the system must be able to measure down to, say, 40% of the original amplitude. This means the shim must be smaller than the optimum, and this

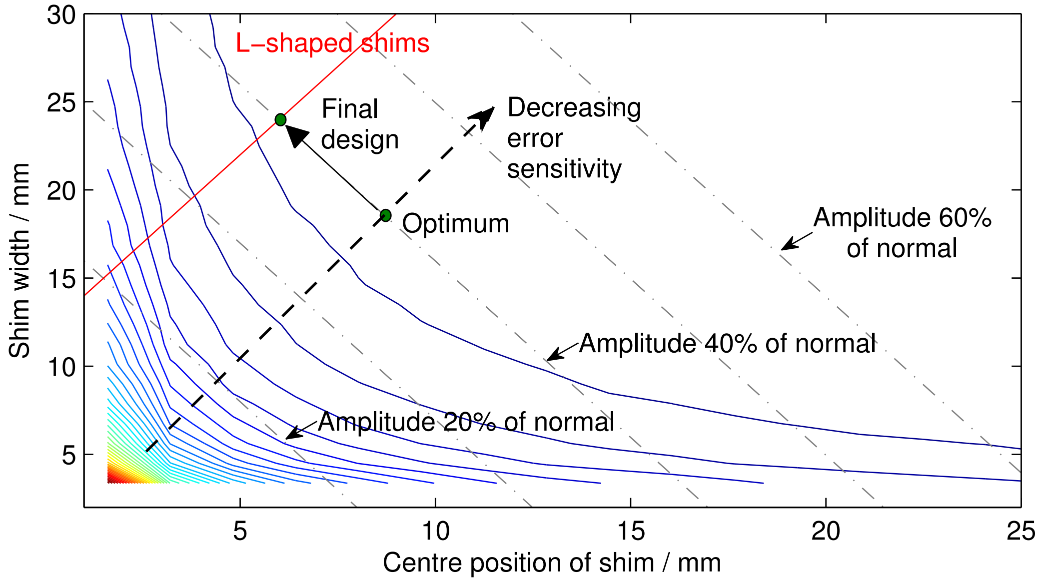


FIGURE 2.3: Shim design: contours of constant error sensitivity, with minimum measurable amplitudes (dashed lines). Final design: width 24mm, centre position 6mm.

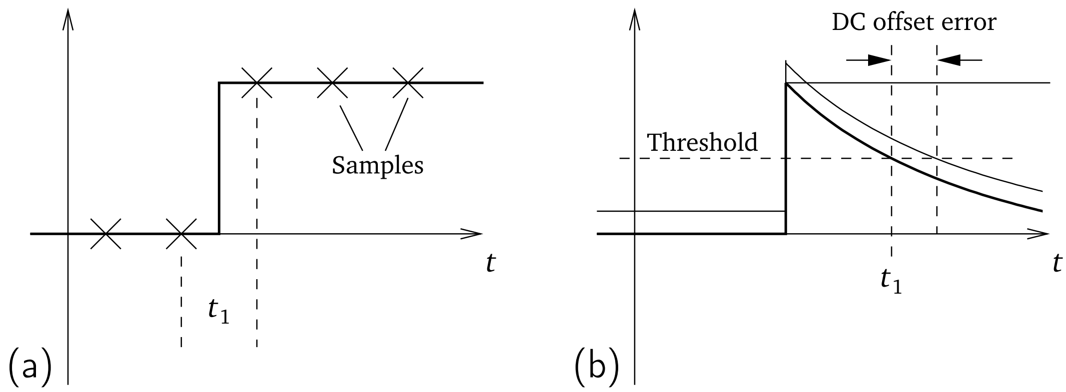


FIGURE 2.4: Locating edges: left, the edge of a square wave can only be found to a resolution of 1 sample. Right, fitting a curve to the exponential decay allows the edge to be found to sub-sample resolution, but any DC offset in the signal must be removed to avoid bias.

constraint is shown by the dashed lines in Figure 2.3.

The final design step was to decide that the simplicity of an L-shaped shim (rather than a more complicated offset rectangle) was worth the small reduction in accuracy shown in Figure 2.3.

2.3 Data acquisition

There are two signals to deal with: the pulse-per-second (PPS) reference from the GPS, and the 4 edges per tick from the IR sensor on the pendulum.

The processing of the input data starts with precisely identifying the times at which the edges in the PPS and IR signals occur. The edge of a square wave (such as these) can only be found to an accuracy of 1 sample, as sketched in Figure 2.4(a). To attain better accuracy, the signals are high-pass filtered to produce a sharp edge with an exponentially decaying tail (Figure 2.4(b); for details of the interface electronics see Appendix A). A curve can then be fitted to this tail, and the time of the edge found by interpolation to sub-sample accuracy. However, the downside to this method is that it is susceptible to variations in DC offset, which cause a shift in the time identified for the edge; care is needed to remove the DC offset for each edge.

A PC datalogger (National Instruments NI-DAQ 6023E) is used to digitise the signals.

2.4 Software processing

The starting point for this project was the work of various previous students, resulting in a MATLAB program which acquires data through the PC's sound card and analyses it to give the current drift (clock time minus true time) and pendulum amplitude. Development this year has built on the same underlying algorithm, but the software has been rewritten entirely in C++, which is more reliable and uses fewer computer resources than MATLAB (not to mention avoiding the need for a MATLAB license). Reliability is much improved, and several problems with the original algorithm which created spurious results have been fixed. The move away from MATLAB also opens up the possibility of removing the PC and running the software on a stand-alone microprocessor. A further improvement was to log data from every tick (every 3 seconds, down from every 15 seconds).

2.4.1 Algorithm

1. Acquire ~3 seconds of data. The datalogger driver records data continuously, and the length of chunk extracted from the acquisition buffer is varied slightly around 3 seconds to keep the chunk aligned with the clock's ticking.

2. Locate the edges in each channel (IR and PPS):

    a) Locate the edges approximately by looking for crossings of a threshold;

    b) For each edge, subtract the mean DC offset over a short period before the edge;

    c) Refine the edge locations to sub-sample accuracy by fitting an exponential curve to the decaying signal. The edge location is taken where the fitted curve crosses a threshold.

3. Calculate drift:

    a) For consistency, drift is always measured relative to the same point in the pendulum's swing: the first edge of the shim as the pendulum swings left-to-right. The correct IR edge can be determined because the gaps between edges are unequal due to the asymmetry of the shim (see example edges in Figure 2.2, where the relevant edge is the first one);

b) Find is the time difference between this reference edge and the previous PPS edge;

c) This time difference must be 'unwrapped' to give drift (because there is a pulse every second, it is only possible to determine drift to modulo 1 second):

   i. if there is a sudden decrease of more than 0.5 seconds relative to the drift from the previous tick, it is assumed than the time difference has wrapped and 1 second is added;

   ii. vica versa for an *increase* of more than 0.5 seconds.

4. Calculate amplitude:

   a) The period (and hence frequency ω) of the pendulum is found from the mean of the time differences between corresponding IR edges in the current and previous ticks.

   b) Using the times of the 4 IR edges $t_1, \ldots t_4$, work back to find the time when the pendulum crossed the centreline, using the following formula (derived by [2, Appendix 6.5], assuming the pendulum motion is sinusoidal):

$$t_0 = \frac{1}{\omega} \arctan \left(\frac{\sin \omega t_1 - \sin \omega t_2 + \sin \omega t_3 - \sin \omega t_4}{\cos \omega t_1 - \cos \omega t_2 + \cos \omega t_3 - \cos \omega t_4} \right) \tag{2.1}$$

   c) Then the amplitude A is given by:

$$A = \frac{W}{\sin \omega(t_1 - t_0) - \sin \omega(t_2 - t_0)} \tag{2.2}$$

   where W is the angular width of the shim.

5. Write the drift and amplitude to the log file with a timestamp (the number of seconds since midnight on Jan 1, 1970).

2.5 Weather logging

Many effects on the clock are environmental, so it is interesting to measure the atmospheric conditions around the clock. The weather station[2] from the previous project was used, placed in the clock enclosure. A program runs once per minute and logs temperature, humidity and pressure.

The temperature, in particular, varies significantly in different locations around the clock, and depending on whether the sensor receives direct sunshine or not, so care is needed about exactly which temperature is being measured. The final location of the sensor is near the pendulum, out of direct sunlight.

[2] A HeavyWeather WS2300, using the Perl software library 'Device::LaCrosse::WS23xx'.

3 Website interface

To ease the viewing and organisation of the large quantity of data collected, a website for the clock has been set up at `http://www.trin.cam.ac.uk/clock`. A screenshot of the main viewing page is shown in Figure 3.1 (although I would recommend visiting the website itself).

The web interface has been implemented in PHP. The main features available through the interface are:

- Browse graphs of data from one or more days;
- Download data, averaged to a selected sampling rate, with selected quantities included;
- View scatter plots, similar to those presented in Section 6;
- Add corrections to the recorded drift, e.g. when the clock is advanced or put back;
- Annotate interesting events with comments.

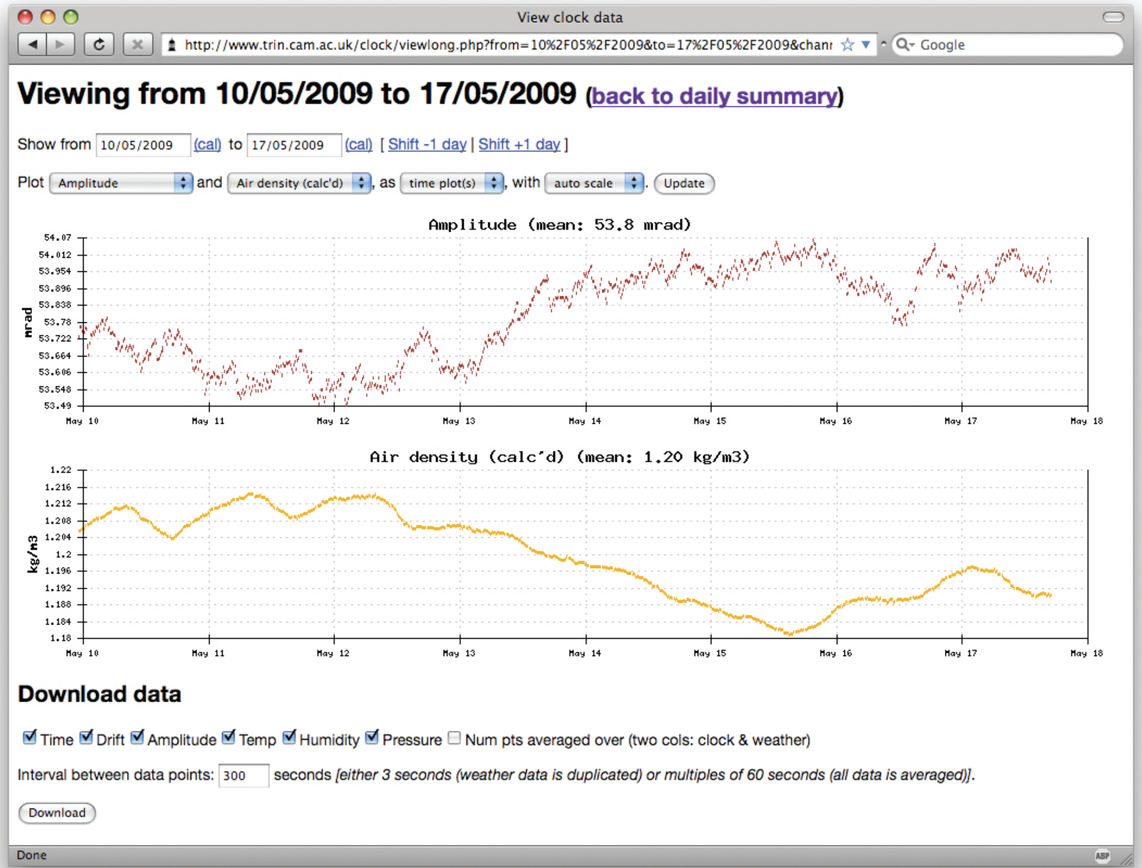


FIGURE 3.1: Screenshot of main web interface for viewing data.

Most of the details of the implementation do not interest us here[1]; only the processing to calculate of air density and going from the raw data are described below.

3.1 Processing

Drift, amplitude, temperature, humidity and pressure are read directly from the log files. Drift and amplitude are sampled at 20 points per minute, the weather data at 1 point per minute; they are usually downsampled for plotting.

Air density is found from temperature T, humidity RH and pressure p as follows [6]:

$$\rho = \frac{p_{\text{dry air}}}{R_d T} + \frac{p_{\text{water vap}}}{R_v T}$$

where $R_d = 287.05$ J/kg K, $R_v = 461.495$ J/kg K, and T is in Kelvin. The water vapour pressure is

$$p_{\text{water vap}} = \text{RH} \times p_{\text{sat}}$$

$$\text{where} \qquad p_{\text{sat}} \approx 610.78 \times 10^{\left(\frac{7.5T - 2048.625}{T - 35.85}\right)}$$

and the dry air pressure is

$$p_{\text{dry air}} = p - p_{\text{water vap}}$$

Going is estimated over N points by

$$G \approx \frac{D_k - D_{k-N}}{t_k - t_{k-N}}$$

where D is drift. Note that there is a trade-off between increased time resolution (small N) and better accuracy in G (large N).

[1] More information about the implementation should be provided on the website in due course.

4 Pendulum analysis

On first sight, a pendulum seems simple: most people who have studied Physics know that its period is given by $T = 2\pi\sqrt{L/g}$. But when we look more closely, it is much more complex. First there is the fairly standard adjustment for non-linearity, which means the period increases as the amplitude of the swing increases. The amplitude in turn depends on the energy dissipated in drag, and so varies with air density. The pendulum is temperature-compensated, which tries to ensure its length is unchanged by thermal expansion; but there are time delays involved which mean transient temperature changes cause the clock to gain or loose time.

The results of these effects can be predicted, and then demonstrated in the recorded data, which is very satisfactory. However, there are more effects that can be predicted from a theoretic starting point (such as tidal variations in gravity) which have not yet been detected in the data. On the other hand there are also events in the data which are yet to be explained.

This section presents the theoretic relationships governing the clock, while the next section presents examples of the recorded data which demonstrate these, and other, effects. The emphasis is on the type of relationship that results, rather than the precise values involved.

4.1 The non-linear pendulum

The equation of motion of a pendulum (Figure 4.1) is

$$mL\ddot{\theta} = -mg\sin\theta$$

For small oscillations, $\sin\theta \approx \theta$, so the equation of motion is linearised to $\ddot{\theta} + (g/L)\theta = 0$. This is standard simple harmonic motion, with period $T_s = 2\pi\sqrt{L/g}$.

When the amplitude of the pendulum A is not negligible, the period is modified, as derived in Appendix B, to be

$$T \approx T_s\left(1 + \frac{A^2}{16}\right) = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{A^2}{16}\right) \tag{4.1}$$

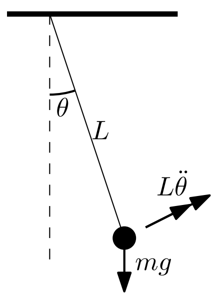


FIGURE 4.1: A simple pendulum

So there are three quantities which affect the period of a pendulum: amplitude, length, and gravity. In the following sections effects which change these quantities will be considered in turn.

4.1.1 Definition of going

In general, we are ultimately most interested in the effect on the clock's going (i.e. seconds gained per second), which is related to pendulum period by

$$G = \frac{T_0 - T}{T_0} = \frac{T_0 - (T_0 + \Delta T)}{T_0} = \frac{-\Delta T}{T_0} \quad (4.2)$$

$$\text{and} \quad \frac{\mathrm{d}G}{\mathrm{d}T} = \frac{-1}{T_0} \quad (4.3)$$

The period T is very close to the nominal period $T_0 = 3$ seconds, so T and T_0 may be used interchangeably in this context. Going is usually given in units of milliseconds per day, with 100 ms/day = 1.16 ppm.

4.2 Basic relationships

The expression for the period of a pendulum (4.1) leads directly to relationships between going, amplitude and length.

If the pendulum has amplitude A, and using the fact that the difference between T & T_s is 4th-order in A and hence negligible,

$$G = \frac{-\Delta T}{T} \approx \frac{-\Delta T}{T_s} = \frac{A^2}{16} \quad (4.4)$$

$$\Longrightarrow \quad \frac{dG}{dA} = -\frac{A}{8} \quad (4.5)$$

Taking the reference amplitude as $A_0 = 54\,\text{mrad}$, this gives $dG/dA = -583\,(\text{ms/day})/\text{mrad}$. Since amplitude is measured directly (unlike length and gravity), this can be directly observed in the data: see Section 6.1.

Similar relationships for L and g will be useful later. Consider the effect of a change in length ΔL:

$$T + \Delta T = 2\pi\sqrt{\frac{L + \Delta L}{g}} \approx 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{2}\cdot\frac{\Delta L}{L}\right]$$

$$\Longrightarrow \quad \frac{\Delta T}{T} = \frac{1}{2}\cdot\frac{\Delta L}{L}$$

Similarly, $\Delta T/T = (-1/2)\Delta g/g$. The following sections discuss physical effects which influence going via these relationships.

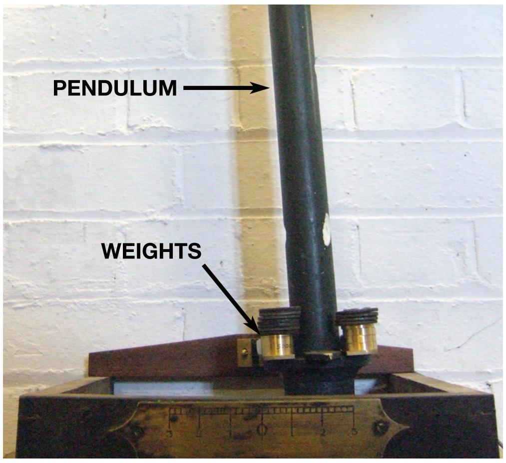


FIGURE 4.2: The regulation weights. The brass weights at the bottom of the stack are calibrated; the washers on top are compensating for a mysterious event on 16 April (Section 6.4).

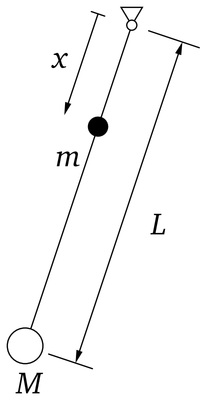


FIGURE 4.3: Simple pendulum with an added weight.

4.3 Mass distribution (regulation)

The clock keeper must be able to adjust the running of the clock, and this is done by adding and removing small weights on a platform halfway down the pendulum (Figure 4.2). The effect of these weights is as follows.

A simple pendulum is shown in Figure 4.3, of length L and bob mass M, with a small mass m added a distance x below the pivot. The kinetic and potential energies are:

$$\begin{aligned} T &= \tfrac{1}{2}mx^2\dot{\theta}^2 + \tfrac{1}{2}ML^2\dot{\theta}^2 \\ V &\approx mgx \cdot \tfrac{\theta^2}{2} + MgL \cdot \tfrac{\theta^2}{2} \end{aligned}$$

(assuming θ is small, so $\cos\theta \approx 1 - \frac{\theta^2}{2}$), so the natural frequency is

$$\omega^2 = \frac{V}{T^\star} = \frac{g(mx + ML)}{mx^2 + ML^2}$$

Putting $\omega_0^2 = g/L$ gives

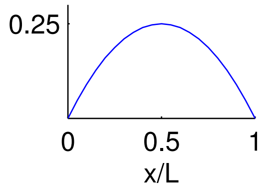
$$\left(\frac{\omega}{\omega_0}\right)^2 = \frac{mx + ML}{mx^2/L + ML} = \frac{1 + \frac{mx}{ML}}{1 + \frac{mx^2}{mL^2}} \approx 1 + \frac{mx}{ML} - \frac{mx^2}{mL^2} \qquad (4.6)$$

The period is $T = 2\pi/\omega$, so

$$\frac{\Delta T}{T_0} = \frac{2\pi/\omega - 2\pi/\omega_0}{2\pi/\omega_0} = \frac{\omega_0}{\omega} - 1$$

which gives (using (4.6) and the binomial expansion)

$$\begin{aligned}\frac{\Delta T}{T_0} &\approx \left[1 - \frac{1}{2}\left(\frac{mx}{mL} - \frac{mx^2}{mL^2}\right)\right] - 1 \\ &= \frac{-m}{2M} \cdot \frac{x}{L}\left(1 - \frac{x}{L}\right)\end{aligned}$$



$f\left(\frac{x}{L}\right) = \frac{x}{L}\left(1 - \frac{x}{L}\right)$ has a maximum at $x/L = 1/2$, so adding weight to the pendulum has the greatest effect when added at the middle — hence the location of the adjustment platform. At the middle, the change in going is

$$\Delta G = \frac{-\Delta T}{T_0} = \frac{1}{8} \cdot \frac{m}{M} \tag{4.7}$$

Since $\Delta T/T = \Delta L/2L$, this has the same effect as a change of pendulum length of

$$\frac{\Delta L}{L} = \frac{-1}{4} \cdot \frac{m}{M}$$

The effect of adding weights to the platform was found by experiment to be 104 ms/day per gram — slightly less than the traditional estimate of 1.2 seconds per day per gram [2]. Incidentally, from (4.7), this means that the mass of the bob must be 104 kg.

4.4 Drag

The pendulum loses energy through air drag and friction, which must be replaced by energy input from the escapement, powered ultimately by the falling weights. The relationship between drag and amplitude can be found from this energy balance.

Drag generally falls in one of two regimes: viscous drag ($\propto V$) and form drag ($\propto V^2$). The Reynolds number *Re* expresses the relative importance of inertial and viscous forces in the flow, and typically viscous drag dominates for flows where *Re* is order 1 or less, while form drag dominates when *Re* is of order 10^3 to 10^5 [3]. We will assume that most of the energy is dissipated in drag around the large bob at the bottom of the pendulum, where the Reynolds number is estimated as 3000.

Assume that the form drag force may be written as $F = \frac{1}{2}\rho V^2 S C_D$, where S is the cross-sectional area and C_D is a drag coefficient. For sinusoidal motion of the pendulum, $\theta = A \sin \omega t$ and the velocity of the bob, at a distance L from the pivot, is $V = LA\omega \cos \omega t$.

The power dissipated is

$$\begin{aligned}P = F \times V &= \frac{1}{2}\rho \left|V^3\right| S C_D \\ &= \frac{1}{2}\rho L^3 A^3 \omega^3 \left|\cos^3 \omega t\right| S C_D\end{aligned}$$

The mean power loss is given by

$$\begin{aligned}\overline{P} &= \frac{1}{2}\rho L^3 A^3 \omega^3 S C_D \cdot \frac{1}{T}\int_0^T \left|\cos^3 \omega t\right| dt \\ &= k\rho L^3 A^3 \omega^3\end{aligned}$$

where k is a constant, dependant on the geometry of the pendulum. This drag loss is balanced by energy input into the pendulum by the escapement, which is in the form of discrete 'kicks', so it is reasonable to assume that the energy per cycle is constant, i.e. $\bar{P}T = 2\pi\bar{P}/\omega = \text{const}$. So

$$\rho L^3 A^3 \omega^2 = \text{const} \tag{4.8}$$

We can derive several results from this, by varying either air density or pendulum length.

4.4.1 Changes in air density

What is the variation in amplitude caused by a change in air density? We know (from Appendix B) that $\omega^2 = \omega_0^2(1 - A^2/8)$. Assuming length L remains constant, substituting this into (4.8) gives

$$\rho A^3\left(1 - \frac{A^2}{8}\right) \approx \rho A^3 = \text{const}$$

$$\therefore \quad \frac{dA}{d\rho} = \frac{-A}{3\rho}$$

At a typical amplitude of 54 milliradians and with a typical air density of $1.2\,\text{kg}\,\text{m}^{-3}$, this gives $dA/d\rho = -15\,\text{mrad}/\text{kg}\,\text{m}^{-3}$. Combining this with dG/dA from (4.5) we get

$$\begin{aligned}\frac{dG}{d\rho} &= \frac{dG}{dA} \times \frac{dA}{d\rho} = \left(\frac{-A}{8}\right) \times \left(\frac{-A}{3\rho}\right) = \frac{A^2}{24\rho} \\ &= 8700\ \text{ms}\,\text{day}^{-1}/\text{kg}\,\text{m}^{-3}\end{aligned}$$

4.4.2 Changes in length

If instead length varies and density is held constant, substituting $\omega^2 = g/L$ into (4.8) gives

$$L^2 A^3 = \text{const}$$

$$\therefore \quad \frac{\mathrm{d}A}{\mathrm{d}L} = \frac{-2A}{3L}$$

Alternatively, eliminating L using $L^3 = g/\omega^6$ gives

$$A^3\omega^{-4} = C_1 \quad \Longrightarrow \quad A^3 T^4 = C_2$$

$$\therefore \quad \frac{\mathrm{d}T}{\mathrm{d}A} = \frac{-3T}{4A} \quad \Longrightarrow \quad \frac{\mathrm{d}G}{\mathrm{d}A} = \frac{-3}{4A}$$

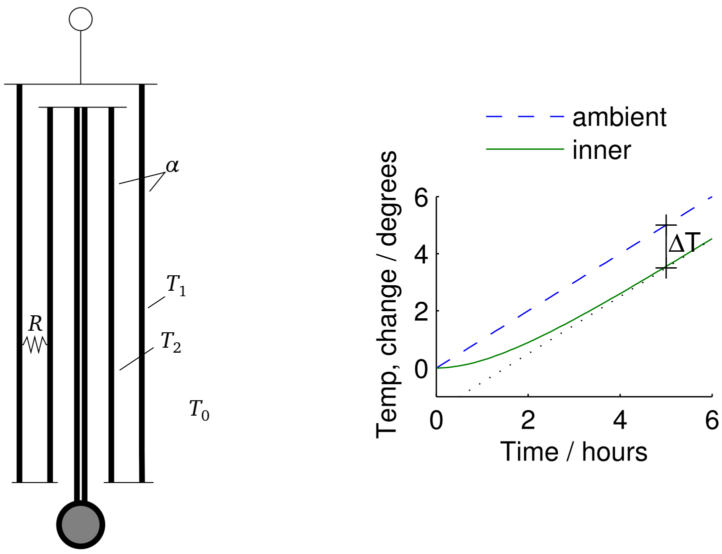


FIGURE 4.4: Simplified temperature-compensated pendulum (left), and response to steadily increasing ambient temperature (right).

Combining these two gives (as expected from $T = 2\pi\sqrt{L/g}$)

$$\frac{\mathrm{d}G}{\mathrm{d}L} = \frac{1}{2L}$$

With $A = 54$ mrad and $L = 2.24$ m, this gives[1]

$$\begin{aligned}
\mathrm{d}A/\mathrm{d}L &= -16\,\mathrm{mrad/m},\\
\mathrm{d}G/\mathrm{d}L &= +19300 \times 10^3\ \mathrm{ms\,day^{-1}/m},\text{ and}\\
\mathrm{d}G/\mathrm{d}A &= -1200 \times 10^3\ \mathrm{ms\,day^{-1}/mrad}.
\end{aligned}$$

Note that this $\mathrm{d}G/\mathrm{d}A$ is enormous; 2400 times larger than the relationship found above in Section 4.2. This value applies only for changes in length, so it is really just saying that length affects going much more than amplitude.

4.5 Temperature compensation

The clock has a temperature-compensated pendulum. Temperature-compensation is intended to remove any variation of pendulum length with temperature, and typically consists of an odd number of elements arranged such that their expansion results in zero net change in length (for example, by choosing a material for the middle layer with twice the thermal expansion of the outer and inner layers). A simplified model of a temperature-compensated pendulum is shown in Figure 4.4, consisting of two outer layers of tube with thermal expansion coefficient α, and an inner tube assumed to be of fixed length. The two outer layers are joined by a thermal resistance R.

[1]The pendulum length is difficult to measure, but this value follows from the 3 second period.

The change of length of the pendulum due to thermal expansion is

$$\Delta L = \alpha L(T_1 - T_2)$$

and the going resulting from this is

$$\Delta G = \frac{-\Delta T}{T} = \frac{-1}{2} \cdot \frac{\Delta L}{L} = \frac{-\alpha}{2}(T_1 - T_2)$$

In steady state, the temperature will be uniform throughout the pendulum and it will be at its design length. The behaviour gets more interesting when the ambient temperature is changing. For example, Appendix C derives the response to a steadily increasing ambient temperature: a steady temperature difference will develop between the layers of the pendulum, which remains as long as the ambient temperature keeps increasing. The size of this temperature difference is, from the Appendix,

$$T_1 - T_2 = \tau \frac{dT_1}{dt}$$

This causes a change in going of

$$\Delta G = \frac{-\alpha\tau}{2} \cdot \frac{dT_1}{dt} = -k\frac{dT_1}{dt}$$

where, making various assumptions about heat transfer and capacity, k is estimated as 40 ms/degree. Since $G = G_0 - k\frac{dT_1}{dt}$, where G_0 should ideally be zero if the correct regulation weights have been used, the drift should vary as

$$D = \int G \, \mathrm{d}t = -kT_1 + G_0 t + \text{const}$$

i.e. drift is proportional to temperature.

The system has a time constant, which is estimated as $\tau \approx 1.5$ hours. These results will only hold once the temperature difference has built up, after a few time constants (perhaps 3 hours). So, we should expect to see a variation in drift due to diurnal temperature changes, and also due to longer-term seasonal changes.

4.6 Changes in gravity

So far, looking at the expression for period

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{A^2}{16}\right)$$

we have considered the effect of changes in amplitude A and length L. The remaining variation to consider is changes in gravity g. There are three main influences on this: the Moon and the Sun's varying gravitational pull, which actually changes g; and buoyancy forces and the added inertia of entrained air, which effectively decrease g.

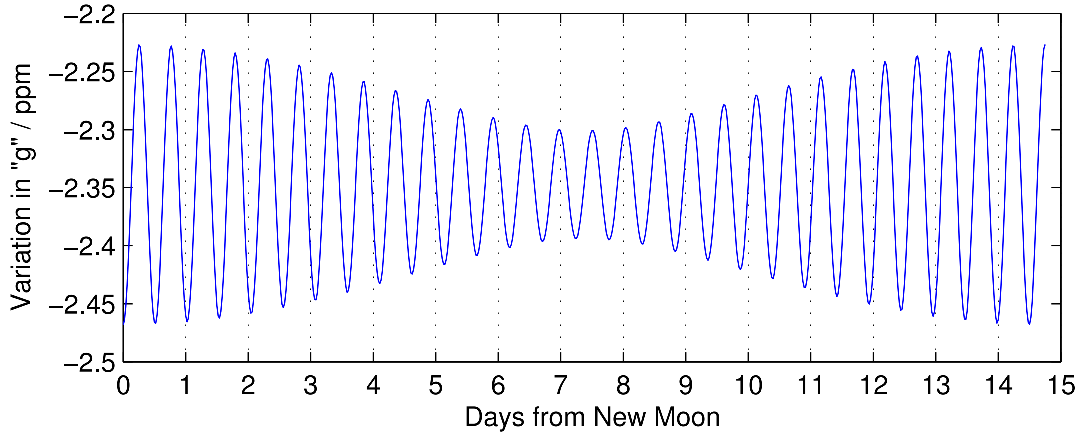


FIGURE 4.5: Variation in g due to Sun and Moon (repeats with period 14.75 days).

4.6.1 Tidal variation

The Moon and the Sun exert a gravitational pull on the Earth, which varies through the year as the Earth orbits the Sun, through the month as the Moon orbits the Earth, and through the day as the Earth spins — their influence is familiar in the form of the tides. The variation in 'g' due to the Moon is given approximately by

$$\frac{\Delta g}{g} = \frac{-1}{2}\left(\frac{R}{r}\right)^3 \left[1 + 2\lambda + 3\lambda\cos 2\phi\right]$$

where R is the radius of the Earth, r is the distance to the Moon, λ is the ratio of the Earth's to the Moon's mass, and ϕ is the longitude of a point on Earth, *relative to the longitude of the Moon* (i.e. $\phi = 0, 2\pi$ when the Moon is overhead). This is valid for points on the equator, and neglects the inclination of the Earth's axis and the declination of the Moon. See Appendix D for the derivation.

This variation occurs twice per 'tidal lunar day,' the period between the Moon passing overhead on successive occasions, which is just under twice per *solar* day. The tidal lunar day a little longer than a solar day (by about 50 minutes), because in the time taken for one rotation of the Earth, the Moon has moved on slightly in its orbit and the Earth must rotate for another 50 minutes before the Moon is again overhead.

There is a similar distinction between the 'sidereal' and 'synodic' lunar months. Although the Moon's sidereal (relative to the background stars) orbital period is 27.3 days, the synodic period (time between successive New Moons) is slightly longer at 29.5 days. This is for the same reason: in the time taken for one complete orbit of the Moon, the Earth has advanced in its orbit around the Sun, and the Moon must move a little further before it is again aligned with the Sun.

The Sun will cause an exactly equivalent variation in g (but a bit smaller). As with tides, where larger 'spring tides' occur twice per month, the interaction of the Sun and the Moon produces a roughly twice-a-day variation in g whose amplitude varies with a period

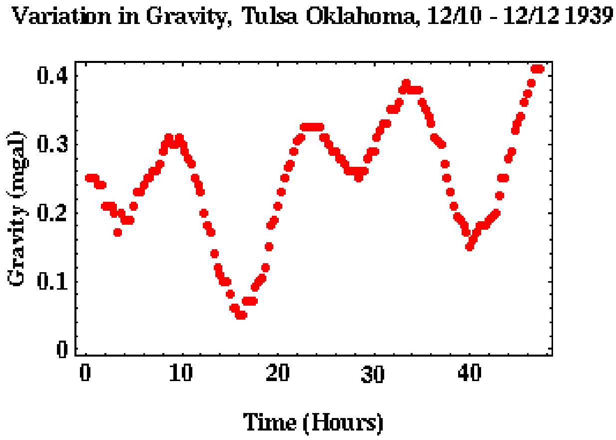


FIGURE 4.6: Gravity measurements from Tusla, Oklahoma, USA (Dec 1939). [7]

of 14.75 days, half the synodic lunar period, as plotted in Figure 4.5.

We expect (Section 4.2) that

$$\frac{\Delta T}{T} = \frac{-1}{2} \cdot \frac{\Delta g}{g}$$

So the change in going resulting from this variation in g is

$$\begin{aligned}\Delta G &= \frac{-\Delta T}{T} \\ &= \pm \left[\frac{3\lambda}{4}\left(\frac{R}{r}\right)^3\right]_{\text{Moon}} \pm \left[\frac{3\lambda}{4}\left(\frac{R}{r}\right)^3\right]_{\text{Sun}} = \pm 3.6 \pm 1.6 \text{ ms / day}\end{aligned}$$

The maximum peak-to-peak variation ('spring tide') is thus 10 milliseconds per day, while the minimum ('neap tide') is 4 milliseconds per day.

To check this value, Figure 4.6 shows some measurements of gravity in Oklahoma from 1939. There is an apparent tidal variation of about ± 0.15 mgal $= \pm 1.5 \times 10^{-6}$ m/s^2. This gives a going of ± 6.6 ms / day.

The derivation above considered points on the equator, whereas Tusla, Oklahoma is at a latitude of 35°N, and so it is closer to the Earth's axis. This will, to a first approximation, reduce R by a factor of $\cos 35°$, and ΔG by a factor of $\cos^3 35°$, which is 55%, to give a predicted peak-to-peak variation in going of between 2.2 and 5.5 ms/day. Furthermore, the inclination of the Earth's axis, which gives rise to the unevenness of the peaks in Figure 4.6, is another important omission from the model of Appendix D. This probably accounts from the remaining discrepancy between the predictions and measurements.

A better estimate would include such details of the geometry of the Moon's motion, but even so the data from Tusla shows that we are quite close to the right answer. The 'right answer' is disappointingly small; preliminary, incorrect, estimates of the tidal effect had predicted a going of the order of ± 300 ms/day. It will be difficult to find such a small variation in the recorded data, but it still should be possible; Section 5 considers what is required.

4.6.2 Buoyancy

The buoyancy of the air around the pendulum will decrease its weight (by Archimedes' Principle), by an amount depending on the air density. The weight of the pendulum is given by

$$W = mg - m_{air}g = mg\left(1 - \frac{m_{air}}{m}\right) = m \cdot g\left(1 - \frac{\rho_{air}}{\rho_{bob}}\right)$$

where m is the mass of the pendulum and m_{air} is the mass of air displaced. This effectively results in a change in gravity of

$$\frac{\Delta g}{g} = \frac{-\rho_{air}}{\rho_{bob}}$$

As before, we expect that

$$\frac{\Delta T}{T} = \frac{-1}{2} \cdot \frac{\Delta g}{g} = \frac{\rho_{air}}{2\rho_{bob}}$$

so the going resulting from variation in ρ_{air} is

$$G = -\frac{\Delta T}{T} = \frac{-\rho_{air}}{2\rho_{bob}}$$

With a steel pendulum bob of density $\rho_{bob} = 7800\ \mathrm{kg/m^3}$,

$$\begin{aligned} \frac{dG}{d\rho_{air}} &= \frac{-1}{2\rho_{bob}} \\ &= -5500\,\mathrm{ms\,day^{-1}/kg\,m^{-3}} \end{aligned}$$

4.6.3 Added mass

As the pendulum swings, air will be entrained and move with the pendulum. This effectively increases the pendulum's inertia without increasing its weight. According to Nelson and Olsson in [3], this effect results in a change of period of

$$\frac{\Delta T}{T_0} = \frac{1}{2}\frac{\kappa m_{air}}{m}$$

where κ is a constant representing the fraction of displaced air m_{air} which acts as added mass. In a perfect, nonviscous fluid in steady-state $\kappa = 1/2$, but in reality the value is higher because the pendulum is accelerating and the fluid has viscosity. A value for κ can be calculated taking these into account (see [3, §C.3]), and for their example pendulum it is $\kappa = 1.18$. For our purposes we will assume $\kappa \sim 1$.

The effect on going is

$$G = \frac{-\kappa}{2} \cdot \frac{\rho_{air}}{\rho_{bob}}$$

$$\therefore \quad \frac{dG}{d\rho_{air}} \approx \frac{-1}{2\rho_{bob}}$$

which is about the same as the buoyancy effect.

4.6.4 Vertical drag

If there was a vertical flow of air past the pendulum, there would be a drag force acting on it which would effectively increase or decrease g. No experimental measurements of air flows around the clock have been made, so this is purely conjecture, but seems possible: the pendulum extends through the floor of the clock room into a chamber in the wall of the Fellow's room below. Since the room below is heated, while the clock room is not, air will tend to flow up or down through the pendulum chamber.

This force can be estimated using the same drag law used in Section 4.4: $F = \frac{1}{2}\rho V^2 S C_D$. An upwards force F will cause an effective change in gravity $\Delta g = \frac{-F}{m}$, where m is the mass of the pendulum bob. To see if this effect might be important, we can find the air velocity needed to cause a change in going of, say, $-100\,\text{ms/day}$:

$$\begin{aligned}\Delta G &= \frac{-\Delta T}{T} = \frac{\Delta g}{2g} = \frac{-F}{2mg} \\ &= \frac{-\rho S C_D}{4mg} V^2\end{aligned}$$

$$\text{so} \quad V = \sqrt{\frac{-4mg\Delta G}{\rho S C_D}}$$

Estimating the area and mass of the bob as 0.03 m^3 and 100 kg (see Section 4.3), and assuming $C_D \approx 1$, this gives $V \approx 0.36\,\text{m/s}$. This is actually quite reasonable; it is possible that this mechanism plays a role but unfortunately without a means of measuring air speed it is not possible to test it.

Without knowing a typical air velocity, because of the V^2 dependence there is no point writing this in the form dG/dV, consistent with the other effects.

4.7 Summary

In summary, the theory above predicts the following effects:

Change in	*affects*	*Relationship*		*Magnitude*
Amplitude	Going	$\frac{dG}{dA} = -\frac{A}{8}$	-583	$\mathrm{ms\,day^{-1}/mrad}$
Air density	Amplitude (drag)	$\frac{dA}{d\rho} = -\frac{A}{3\rho}$	-15	$\mathrm{mrad/kg\,m^{-3}}$
	Going (via amplitude)	$\frac{dG}{d\rho} = \frac{A^2}{24\rho}$	8700	
	(buoyancy)	$\frac{dG}{d\rho} = \frac{-1}{2g\rho_{pend}}$	-5500	
	(added mass)	$\frac{dG}{d\rho} = \frac{-1}{2g\rho_{pend}}$	-5500	
	total		-2300	$\mathrm{ms\,day^{-1}/kg\,m^{-3}}$
Length	Amplitude	$\frac{dA}{dL} = \frac{-2A}{3L}$	-16	$\mathrm{mrad/m}$
	Going	$\frac{dG}{dL} = \frac{1}{2L}$	19.3×10^6	$\mathrm{ms\,day^{-1}/m}$
	hence	$\frac{dG}{dA} = \frac{-3}{4A}$	-1.2×10^5	$\mathrm{ms\,day^{-1}/mrad}$
Temperature	Drift (temp. comp'n)	$\Delta D = -kT_{amb}$	-40	ms/°C
Tidal effects	Going (gravity)	$\Delta G = \pm\frac{-3\lambda}{2}\left(\frac{R}{r}\right)^3$	± 10	ms/day (max)
Draughts	Going (vert. drag)	$\Delta G = \frac{-\rho S C_D}{4mg} V^2$		

4.7.1 Properties of air

Since it is not density but pressure, temperature and humidity that is measured, it is convenient to work in terms of those quantities. In the conditions experienced in the clock tower, the major influence on density is pressure; Figure 6.4 justifies this assertion. Assuming this is the only influence, from the ideal gas law $p = \rho RT$ we get

$$\frac{\mathrm{d}\rho}{\mathrm{d}p} = \frac{1}{RT}$$

thus, for example,

$$\frac{dA}{dp} = \frac{dA}{d\rho} \times \frac{d\rho}{dp} = \frac{-A}{3\rho \cdot RT} = \frac{-A}{3p}$$

$$\text{and} \qquad \frac{dG}{dp} = \frac{A^2}{24\rho RT} = \frac{A^2}{24p}$$

At a typical pressure of 1010 hPa, this gives $dA/dp = -0.018\,\mathrm{mrad/hPa}$ and $dG/dp = 0.12\,\mathrm{ms\,day^{-1}/hPa}$.

5 Extracting tidal variations in gravity

We saw in Section 4.6.1 that the tidal variation in gravity will cause a periodic change in going of at most ±10 milliseconds per day. This is tiny compared to the noise in the measurements of going, which has an RMS value of about 600 ms/day. Nonetheless spectral analysis, given a sufficiently long period of data, should be able to discern this periodic variation from the noise. But how long is 'sufficiently long'?

The first condition is related to the duration of the data. With a sample rate F_s and data of duration T, the number of samples is $N = F_s T$. The number of points in the spectrum is also N, so the bin width $\Delta f = F_s/N = 1/T$. The tidal variation occurs at a frequency of approximately 2 cycles per day, the frequency bins must be sufficiently narrow for this to be distinguishable from lower frequency variations. For example, if we want to find the tidal variation in the 10th bin, we need $\Delta f = 0.2$ cycles per day. This means T must be at least 5 days.

The second condition is that the tidal variation be visible above the noise. 'White' noise has a flat spectrum (Figure 5.1), and it is a property of spectra that the area under the spectrum is equal to the mean square (MS) of the time-domain signal. If the maximum frequency content of the noise is F_N and the amplitude of the spectrum is B^2 then

$$\text{Area under spectrum} = 2 \times 2\pi F_N \times B^2 = \mathrm{MS}_{\text{noise}}$$
$$\implies \quad B^2 = \frac{\mathrm{MS}_{\text{noise}}}{4\pi F_N}$$

The Fourier transform of the periodic tidal variation (with RMS value A) is $\frac{A^2}{2}\delta(\omega + \omega_0) + \frac{A^2}{2}\delta(\omega - \omega_0)$. In a real spectrum, we cannot have infinitely thin δ-functions, so the best approximation will be that the spike falls into only one frequency bin of width $\Delta\omega$ and height C^2, say. The areas of the bin and the δ-function must be equal, i.e.

$$C^2 \Delta\omega = \frac{A^2}{2} \int_{-\infty}^{+\infty} \delta(\omega - \omega_0)\,\mathrm{d}\omega = \frac{A^2}{2}$$

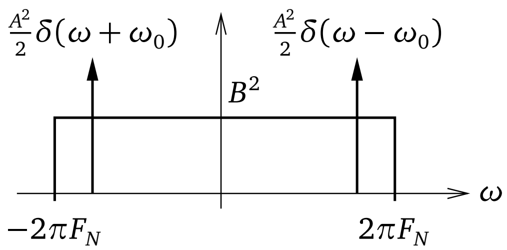


FIGURE 5.1: Spectrum of $A\sqrt{2}\cos\omega t$ and white noise, RMS $= B^2$

$$\implies \quad C^2 = \frac{A^2}{2\Delta\omega} = \frac{A^2}{4\pi} \cdot \frac{1}{\Delta f} = \frac{A^2 T}{4\pi}$$

In practical spectral analysis, a window is used to reduce the effect of having a finite-length non-periodic signal. This reduces spectral leakage, at the expense of reducing the peak height of a spike. For example, the Hann window reduces the peak value by one half. So be able to discern the tidal variation in the spectrum above the noise, and including a factor of one half to compensate for the window, we need

$$\begin{aligned} C^2 &> 2 \times B^2 \\ \frac{A^2 T}{4\pi} &> \frac{2 \times \text{MS}_{\text{noise}}}{4\pi F_N} \\ T &> \frac{2 \times \text{MS}_{\text{noise}}}{A^2 F_N} \end{aligned}$$

or, putting $T = N/F_s$,

$$N > \frac{2 \times \text{MS}_{\text{noise}}}{A^2} \cdot \frac{F_s}{F_N}$$

If the noise has a wide frequency range, we could assume it has frequency content right up to the Nyquist frequency, i.e. $F_N = F_{nyq} = F_s/2$. Then

$$N > \frac{4 \times \text{MS}_{\text{noise}}}{A^2}$$

Putting values into this, the maximum RMS value of the tidal variation is $A = \frac{10}{\sqrt{2}} =$ 7 ms/day, and that of the noise is 600 ms/day. So we need

$$N > \frac{4 \times 600^2}{7^2} = 3 \times 10^4$$

which, at a sample rate of once every 3 seconds, is about 1 day. When the tidal variation is at its minimum of about ±4 ms/day, the requirement is increased to $N > 9 \times 10^4$, or 3 days.

Next is the question of accuracy. Even if the noise was roughly 'white,' its spectrum will itself be noisy because we are sampling it over a finite period. Newland [8, Ch.9] shows that the expected accuracy of a real spectrum is

$$\frac{\sigma}{m} \approx \frac{1}{\sqrt{B_e T}}$$

where m and σ are the mean and standard deviation of the spectrum, B_e is the effective bandwidth of the spectral window and T is the record length, as above. The effective bandwidth depends on the shape of the window which is used, but it is given roughly by $B_e \approx 1/T$. T can be increased without increasing B_e, to get better accuracy, by averaging. So, to obtain an accuracy of say $\sigma/m = 0.1$, we need $B_e T = 100$, i.e. 100-fold averaging.

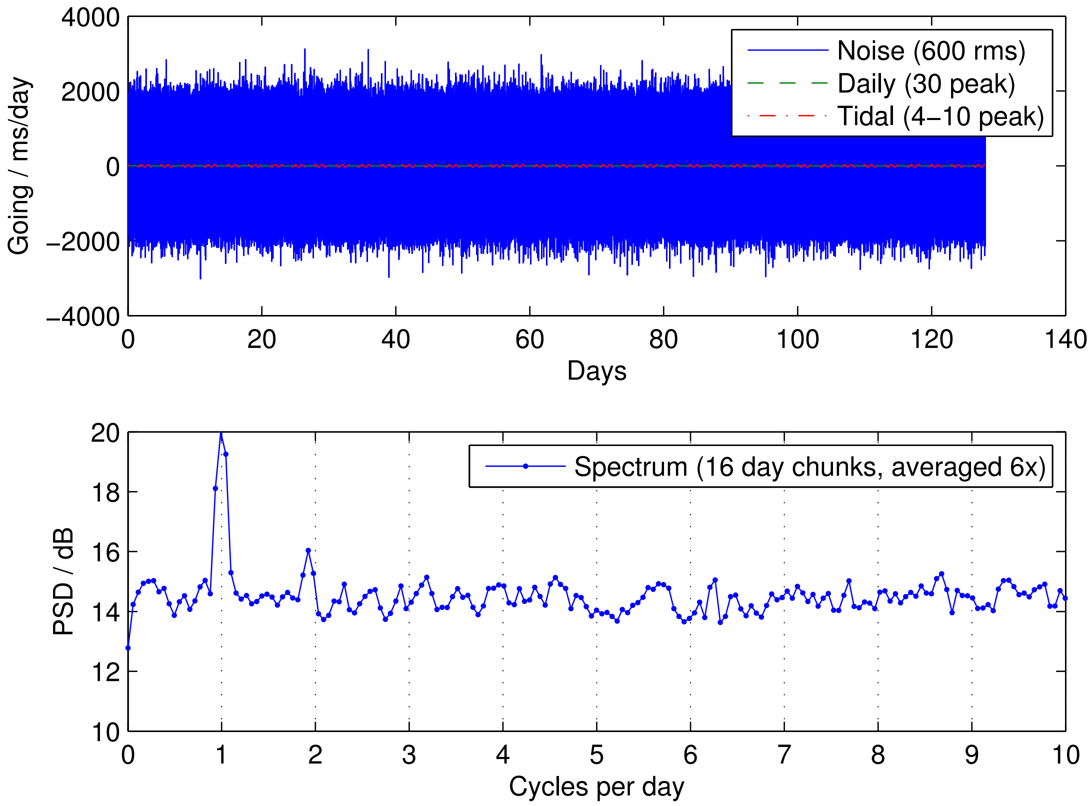


FIGURE 5.2: Spectral analysis of simulated tidal variation.

With $B_e = 0.2$ cycles per day as above, this means we require $T = 500$ days. This is clearly quite a long time.

To get satisfactory results with less data, we could either require less accuracy or reduce the level of the noise. An accuracy of 10% is probably more than sufficient to detect the tidal variation, so we could settle for 33%, giving $B_e T = 9$ and $T = 45$ days — this is much more manageable.

The noise level can be reduced by calculating the going over a longer interval, but at the expense of a reduced sample rate. For example, calculating the going every 30 seconds instead of 3 seconds gives an RMS noise of about 280 ms / day. Thus the number of points needed is

$$N > \frac{4 \times 280^2}{7^2} = 6400$$

which is 2.2 days. So increasing the interval between data points requires a *longer* period of data, even though the noise level is decreased.

5.1 Simulation

Simulated noise and tidal signals were used to do some virtual experimentation to check the above results. The RMS value of the noise in going was estimated as 600 ms / day, and a tidal effect of between 4 and 10 ms / day peak was added in, along with a generic daily variation. There are many combinations of possible windows, data lengths and averaging, but the one shown in Figure 5.2 seems to give good results from minimal data, while still reliably and clearly identifying the tidal signal: this simulation used 96 days of data in 16 day chunks, averaged 6 times, using a Hamming window. This suggests that the calculations above may underestimate the amount of data required to reliably identify the tidal variation.

6 Results & Discussion

6.1 Changes in amplitude

Over the period 3rd–6th May 2009, there were two occasions on which the pendulum amplitude dropped significantly, for no apparent reason. This gives the opportunity to test the relationship derived in Section 4.2 between going and amplitude. The upper part of Figure 6.1 shows the time histories of amplitude and going, while the lower part shows going plotted against amplitude; it is clear that there is a link between them. The dashed line corresponds to the result of Section 4.2, that $dG/dA = -583\,\mathrm{ms\,day^{-1}/mrad}$. The match between this and the data seems extremely good.

Another example is shown in Figure 6.2. On this occasion (in the early hours of the 15th April 2009), the clock had not been wound and when the weights reached the bottom of their descent, the clock stopped. In this case, the slope of −583 ms/day/mrad appears to hold initially, but as the pendulum amplitude decays further, going increases faster than predicted.

What might cause this? One potential problem is that for large changes in amplitude,

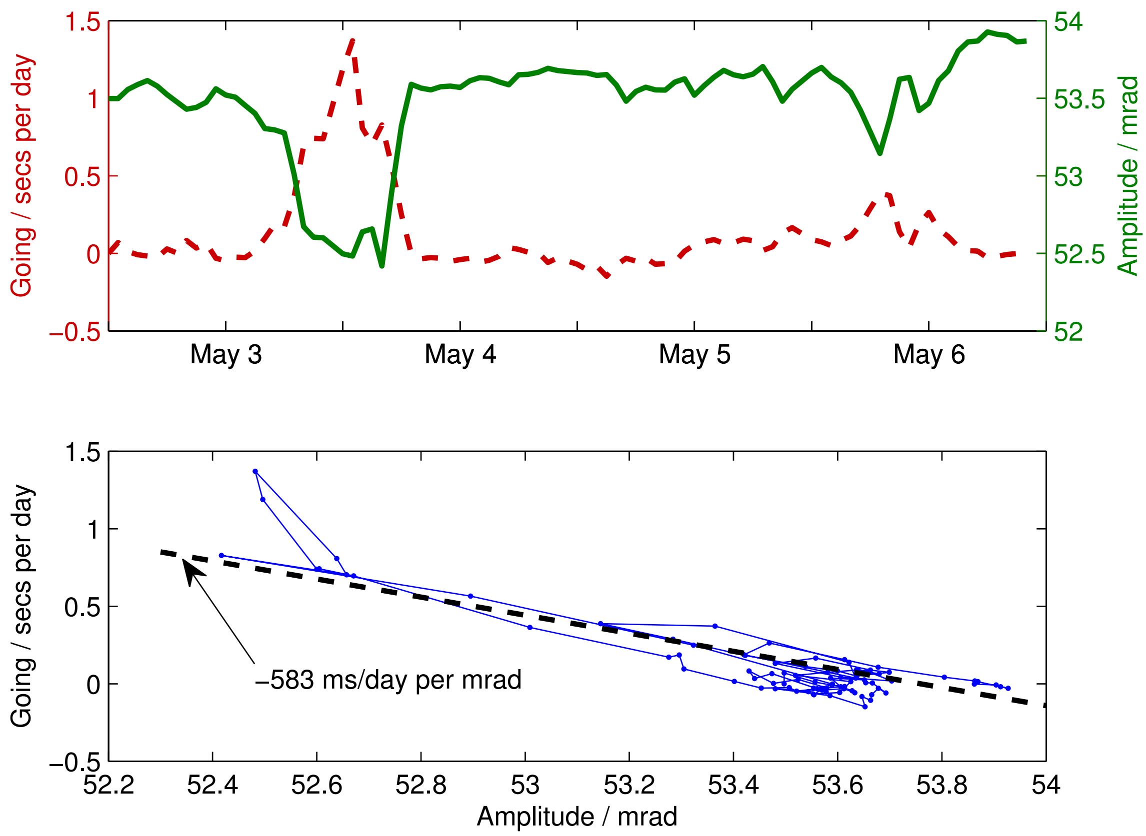


FIGURE 6.1: Going (dashed) & amplitude (solid). [3–6 May 2009, averaged hourly]

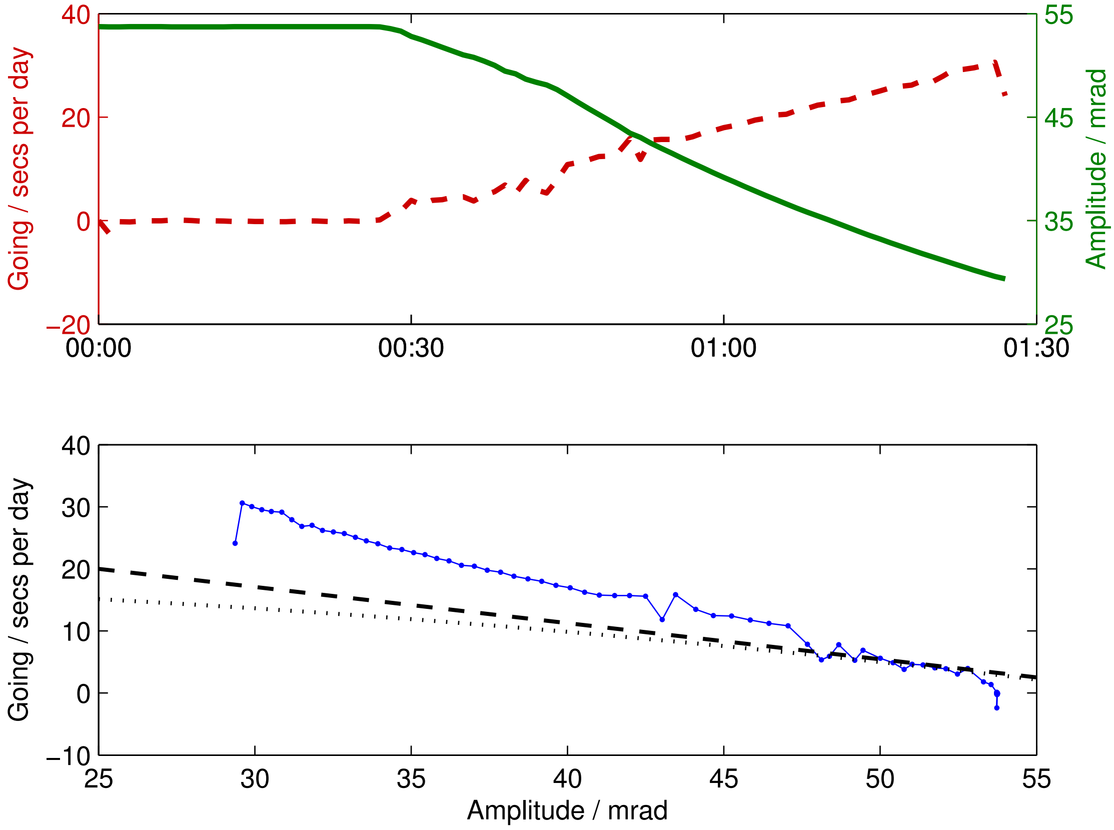


FIGURE 6.2: Going (dashed) & amplitude (solid). [15 April 2009, averaged over 1 minute]

the linear variation used above will no longer hold. Rather than a straight line, we need to plot the curve given by

$$G = \text{const} - \frac{A^2}{16}$$

This is shown in Figure 6.2 by the dotted curve: this improvement does not explain the discrepancy.

Another possibility is that the decay is being disrupted by variations in the weight tension, as the weights settle onto the bottom of their shaft. It would be interesting to repeat this analysis when the clock has been purposefully stopped and the weights cannot influence the pendulum. In any case, there is another effect at work.

Figure 6.3 shows a third example. Here, over the course of 7 days, there seem to be two distinct 'blobs' in which the going-amplitude law is followed, but something else has happened to shift between them. This might indicate a new effect with a *positive* slope of going against amplitude.

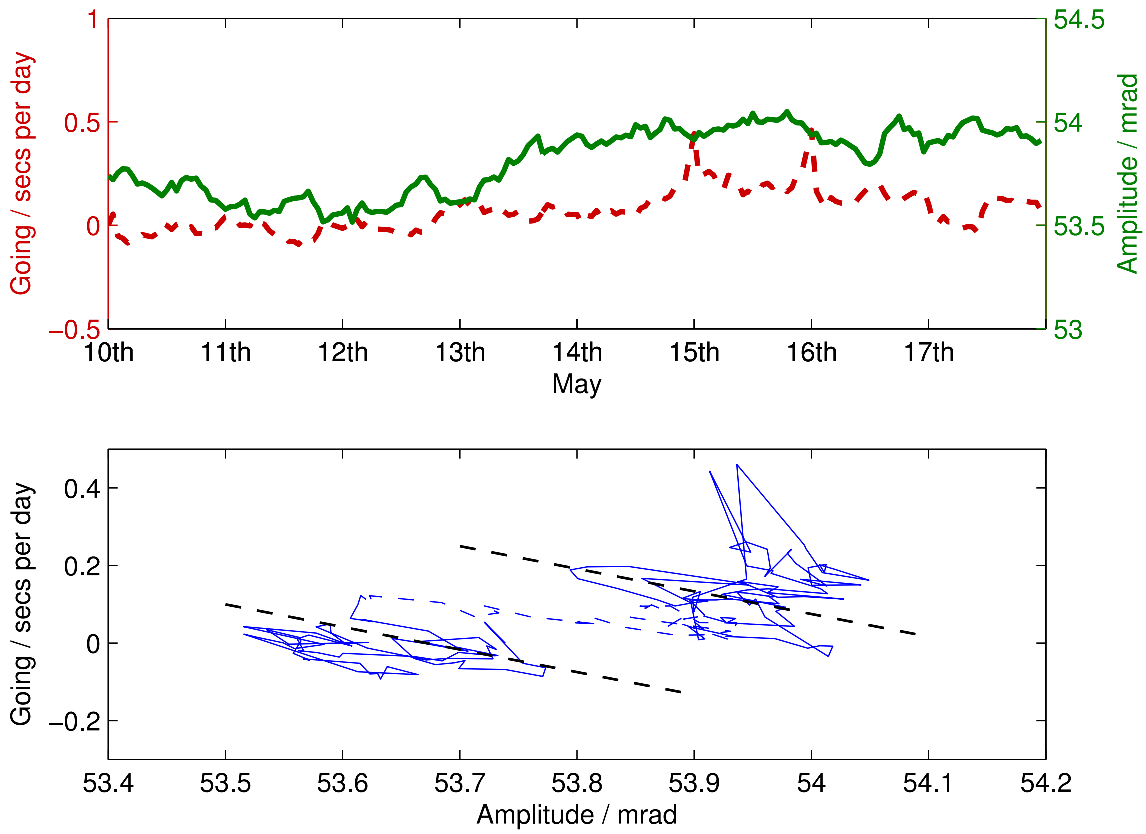


FIGURE 6.3: Going (dashed) & amplitude (solid). [10–17 May 2009, averaged hourly]

6.2 Changes in air density

As discussed in Section 4.7.1, pressure is a good proxy for air density, and is used in this example; Figure 6.4 shows the correlation between density, pressure, humidity and temperature to justify this. Figure 6.5 shows the variation of pressure and amplitude over 27 days in April & May 2009. As before, the upper part shows the time histories, and the lower part shows amplitude plotted against pressure.

The theoretical line of slope $dA/dp = -0.018$ mrad / hPa is drawn, and the agreement with the data is quite good. Note that there are two outliers, which correspond to the unusual drop in amplitude observed on the 3rd & 4th May and discussed in Section 6.1.

Overall we expect going to decrease with increasing air density (from the combined effects of drag, buoyancy and added mass) at $-2300\,\mathrm{ms\,day^{-1}/kg\,m^{-3}}$. The going over the same period as above is shown in Figure 6.6. There seem to be two periods where the going—density relationship is similar to predictions, separated by the large drop in amplitude on the 3rd & 4th May, although the gradient is closer to -6000 (dotted line) than -2300 (dashed line). As there are three effects contributing here it is difficult to know which prediction is wrong.

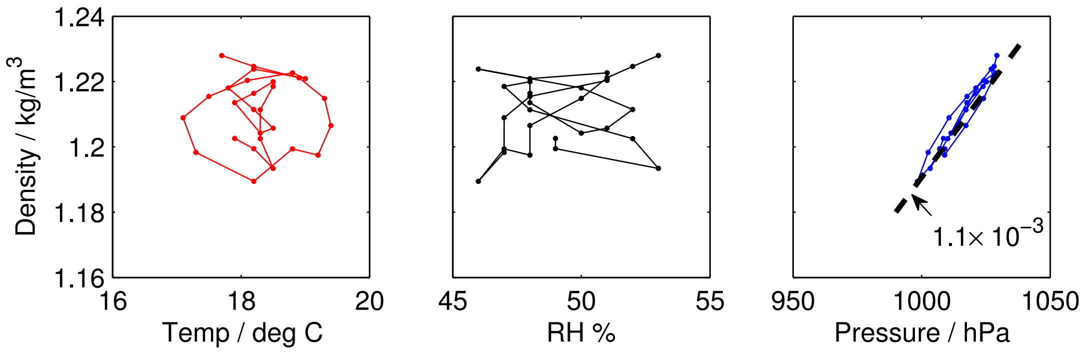


FIGURE 6.4: Density against air temperature, humidity and pressure. Pressure is the dominant influence. [20 April–17 May 2009, averaged daily].

6.3 Temperature & drift

Figure 6.7 shows the relationship between drift and temperature. The best-fit value is about −150 ms / degree, which is rather more than the value of −40 found in Section 4.5. This difference is unsurprising, given the many assumptions made about heat capacity, internal surface heat resistance and internal dimensions of the pendulum in order to find the value of −40.

With the predicted time constant of about 1.5 hours, there should be changes in drift due to short-term variations in temperature as well as the longer term ones just considered. Figure 6.8 shows the same period as Figure 6.7, but at a sample rate of once per hour rather than once per day. The same overall correlation is visible, but the diurnal temperature swings do not seem to have the expected effect on the drift. This suggests that the time constant is actually somewhat longer than 1.5 hours: either the thermal mass of the pendulum layers or the thermal resistance between them is larger than estimated.

6.4 Changes in pendulum length

Step changes in the length of the pendulum are not usually expected in a clock, apart from the temperature compensation effects discussed above. However, there was one occasion when this may have happened. The clock was not wound when it should have been, and it stopped in the early hours of 15 April 2009 (Figure 6.9, left). When it was restarted, the going had dropped to −23.5 s/day (Figure 6.9, right). This could have been due to the pendulum knocking into the wall and somehow making the bob weight fall slightly, as it was pushed to restart it. There are scratches on the IR sensor fitting to support this suggestion, but it is just a theory.

In any case, the change in going caused by a change in length is

$$\Delta G = \frac{-\Delta T}{T} = \frac{-\Delta L}{2L}$$

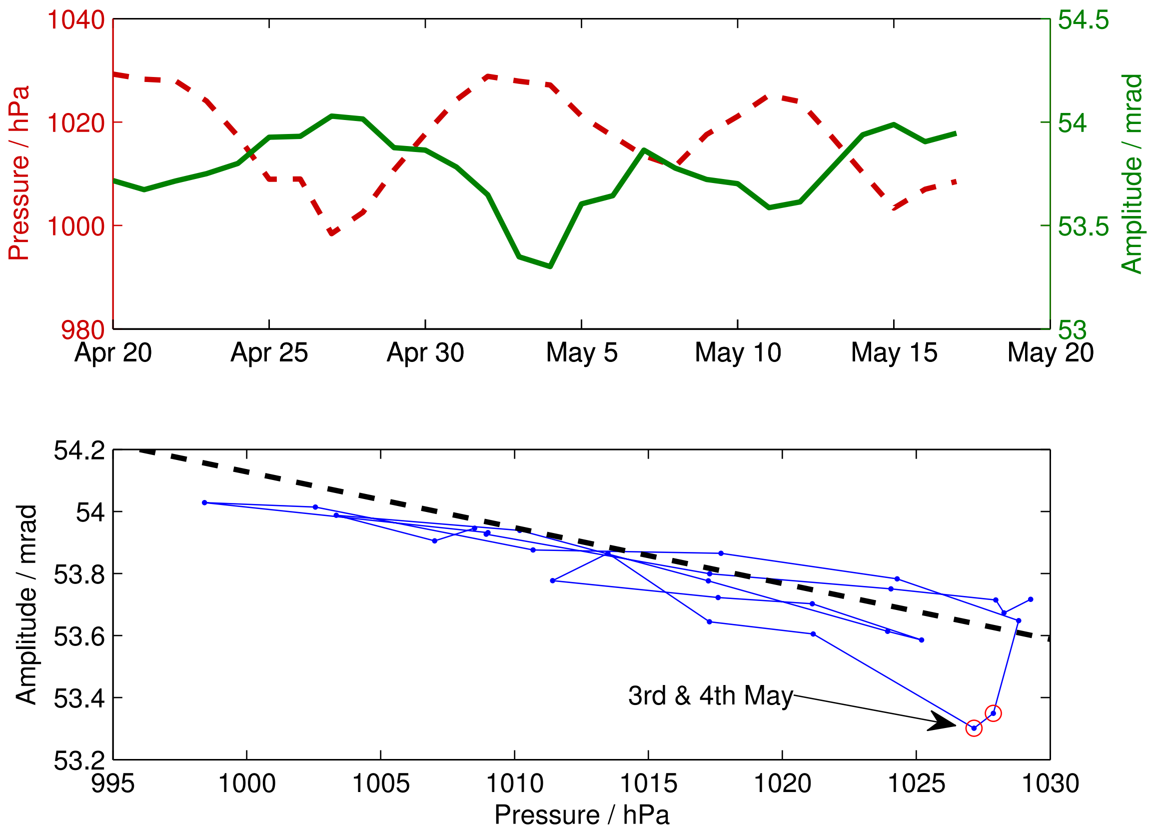


FIGURE 6.5: Amplitude (solid) & air pressure (dashed). [20 April–17 May 2009, averaged daily]. The points labelled "3rd & 4th May" are at unusually low amplitude, see Figure 6.1.

so this change in going would require a increase in length of 1.2 mm (given that the length of the pendulum is 2.2 m).

The going in Figure 6.9 returns to normal with the addition of lots of washers, which can be seen in the photo in Figure 4.2.

6.5 Tidal variations

The theory of extracting the very small tidal variations from everything else that is going on was discussed in Section 5, with the conclusion that at least 45 days of data would be required, probably more. Unfortunately currently only about 30 days of uninterrupted data are available; nonetheless it has been analysed and is shown in Figure 6.10. Here, the spectrum of the going has been found, both at the original sample interval of 3 seconds, and also calculated over one-hour intervals (only the hourly going is shown in the time domain; the noise in the 3-second going would swamp the plot). The spectra are shown below, for both the 3 second and hourly data. A Hann window of length of 5 or 10 'tidal lunar days' (i.e. period between the Moon passing overhead on successive occasions, slightly more than a solar day) was used to try and match the periodicy of the signal.

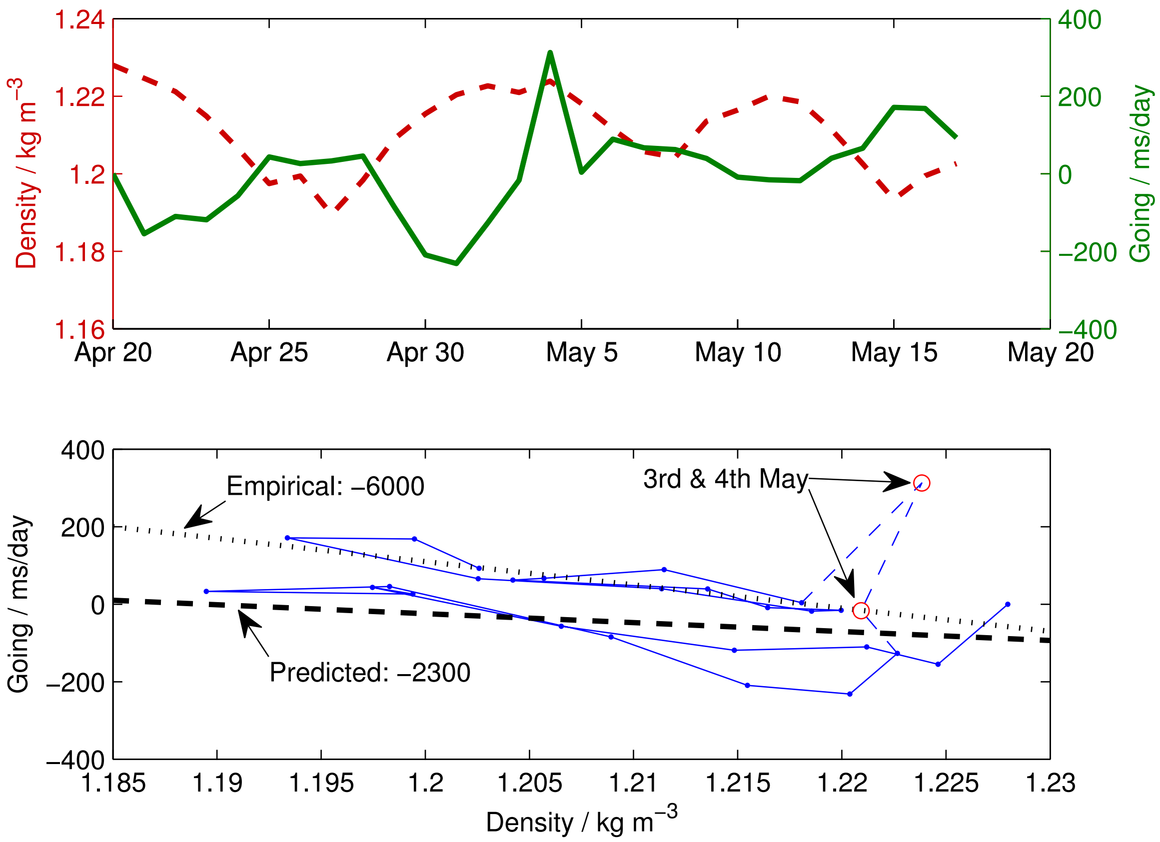


FIGURE 6.6: Going (solid) & air density (dashed). [20 April–17 May 2009, averaged daily]. As above the points labelled "3rd & 4th May" are outliers.

No tidal variation is visible; although there might seem to be a small peak at 2 cycles per day, this is still close to the noise level and in any case does not appear when the 10 day window is used. There still seems to be quite a lot of spectral leakage raising the spectrum at low frequencies, which could perhaps be reduced by clever use of other spectral analysis techniques. Hopefully also once a longer complete data record is available, better results will be possible.

It may also be possible to improve the going data by removing unwanted variations. For example, changes in amplitude have a large effect on going, but since the amplitude is known at every point in theory this variation could be subtracted, leaving a cleaner going signal.

6.6 Unexplained results

6.6.1 Step changes in going

Figure 6.11 shows an occasion on which the going changed quite significantly (about 150 ms / day) for no obvious reason. The other recorded quantities are also shown, and there

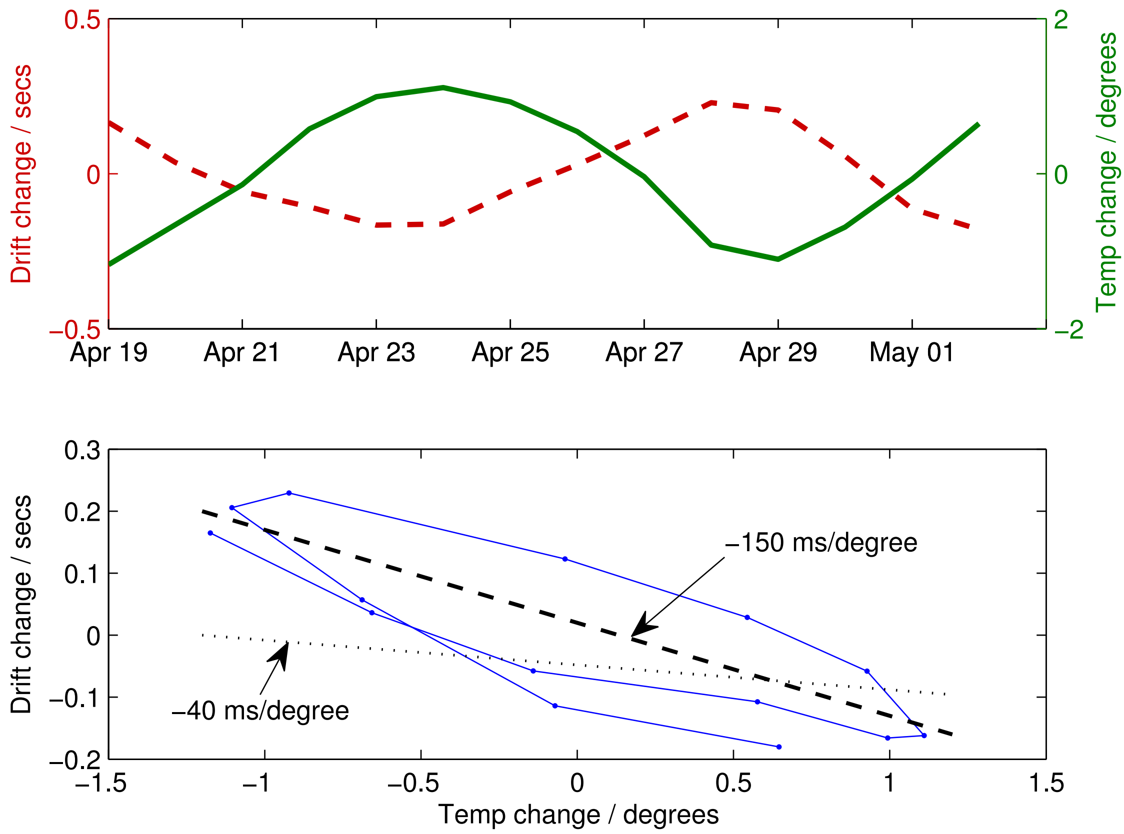


FIGURE 6.7: Drift (dashed) & temperature (solid). [19 April–2 May 2009, averaged daily].

seem to be no changes in them which might have caused it.

One theory is that, since the biggest change occurred roughly at midnight, the clock striking the hour may have prompted the change. Indeed, looking closely at the data around midnight, the going does seems to drop immediately after midnight, although the accuracy in going is quite low at short time scales so it is somewhat vague. A possible mechanism for this is the air currents created during the bells striking, which come from the vanes fitted to the bell-ringing wheels to regulate the speed of striking.

The vertical drag mechanism described in Section 4.6.4 might play a part here. Perhaps the striking of the bells could have reversed the direction of flow through the pendulum chamber, if there was little initial preference for either direction. An air flow sensor would make an interesting addition to the monitoring equipment.

6.6.2 Sudden drop in amplitude

Although the event shown in Figure 6.1 proves very nicely the going-amplitude relationship, there is no explanation in the other data for the sudden large drop in amplitude itself. Perhaps the falling weight snagged on something; this could have reduced the tension for a while, until the weight had descended further and freed itself.

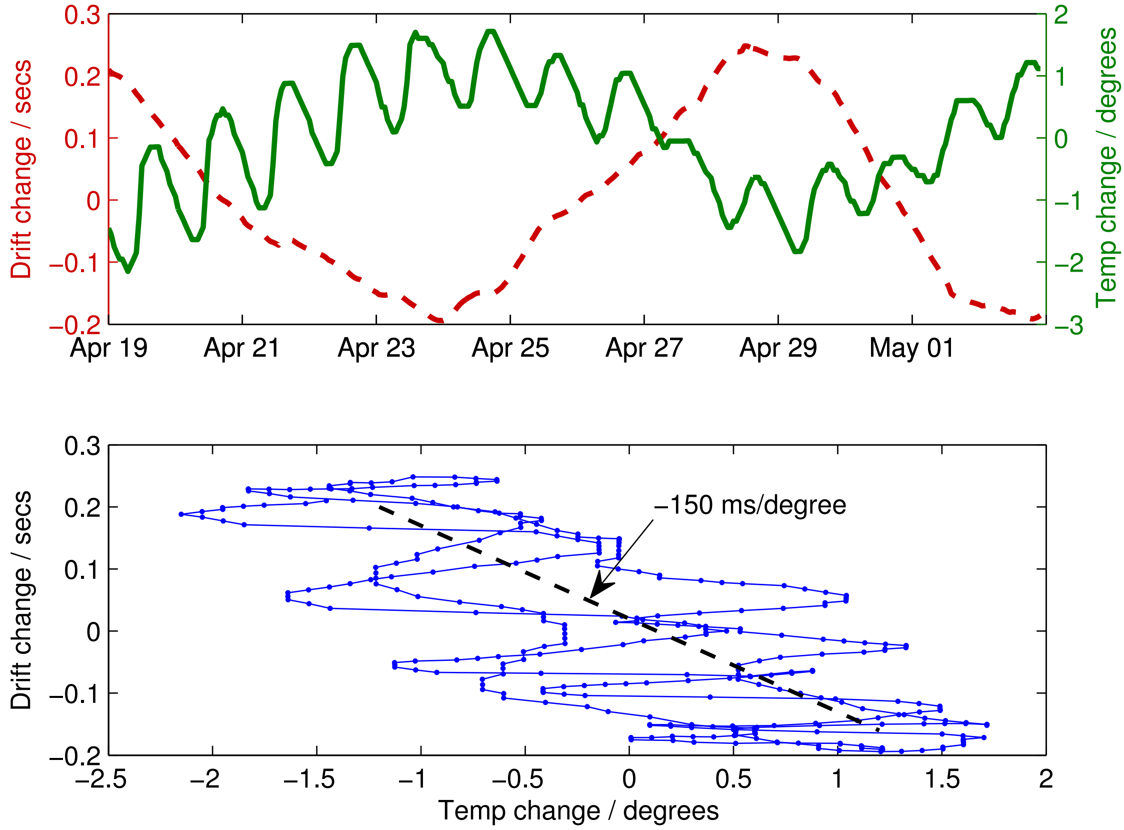


FIGURE 6.8: Drift (dashed) & temperature (solid). [19 April–2 May 2009, averaged hourly].

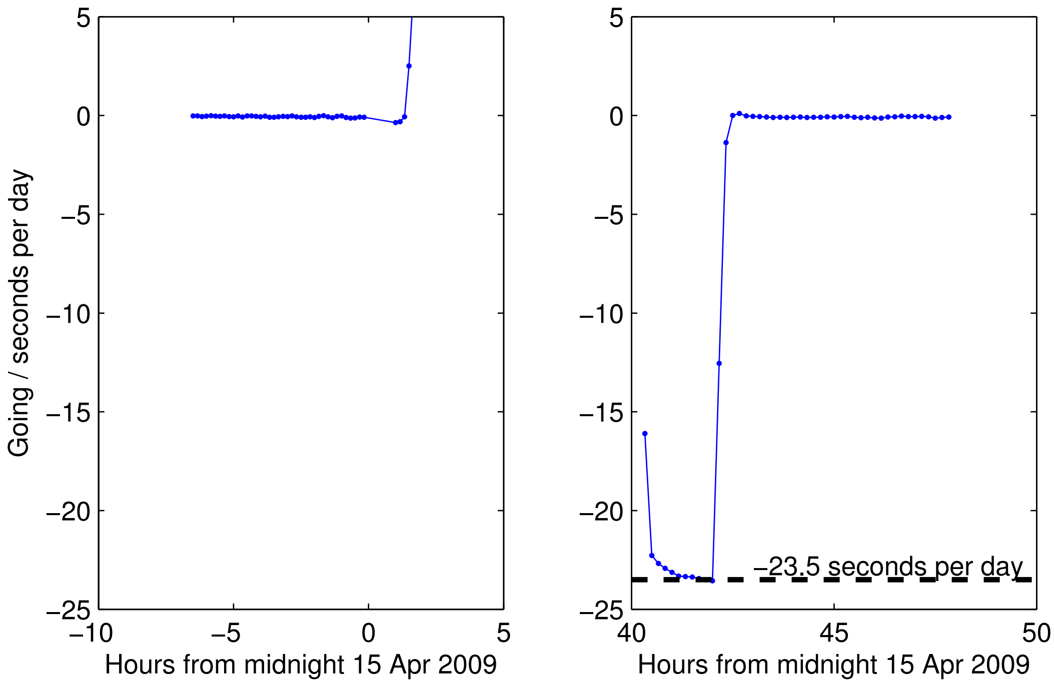


FIGURE 6.9: An accidental change in pendulum length? [14–16 April 2009, averaged over 10 minutes].

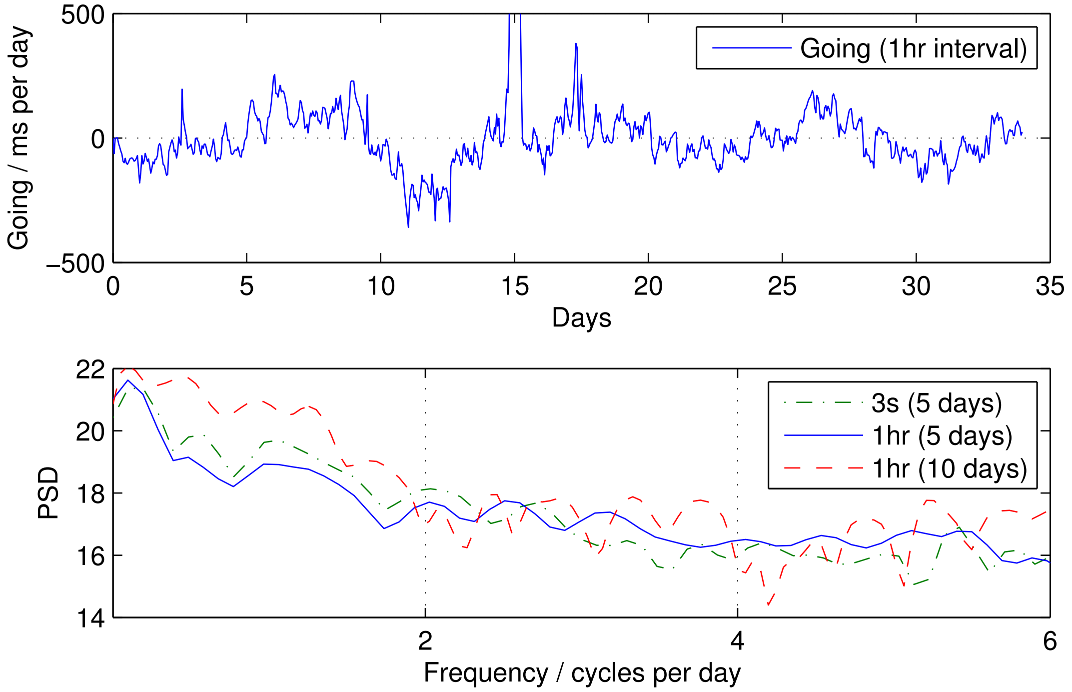


FIGURE 6.10: Spectral analysis: looking for the tidal variation in going. [19 April–22 May].

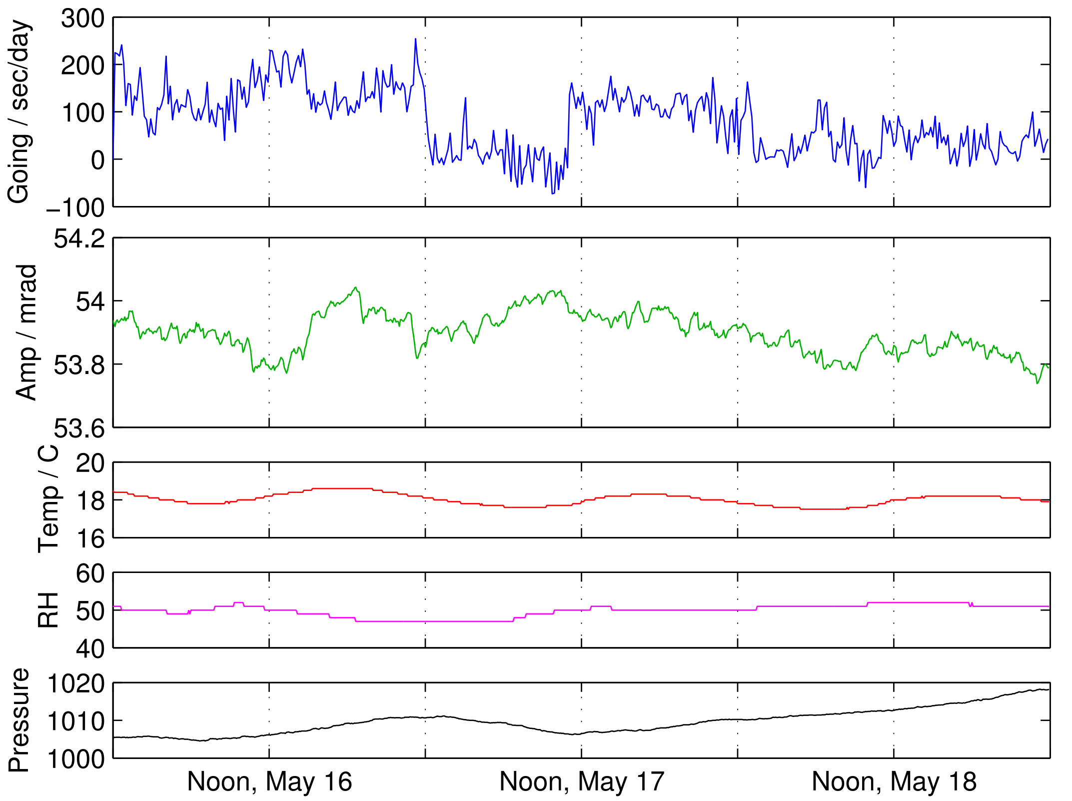


FIGURE 6.11: Unexplained changes in going. [16–18 May 2009]

7 Conclusions

7.1 Clock monitoring

The first part of the project was about producing a reliable monitoring system for the clock. This has been achieved, with a program written in C++ which measures the clock's drift and pendulum amplitude every 3 seconds, with reference to the accurate GPS time. The environmental conditions around the clock have been observed to have a strong influence on its running, so in addition to the measurements of the clock itself, temperature, pressure and humidity in the clock enclosure are logged every minute.

In this process, the need for care in the quality of the data (e.g. removing DC offsets and dealing with noise) was noted, and the software and electronics improved to deal with these issues.

To allow easy use of the recorded data, a website has been set up which allows the data to be browsed, plotted and downloaded.

Finally, using the system to measure the change in going when adding weights to the pendulum, the calibration factor between added mass and going was established, and a set of calibrated adjustment weights were produced.

7.2 Analysis

Theoretical relationships have been derived between the clock's going, pendulum amplitude, mass distribution & length, air temperature & density, the presence of draughts, and tidal variations in gravity.

These predictions were tested against real data from the clock, with a good agreement found for variations in amplitude, mass distribution, air temperature and air density. It was not possible to measure the pendulum length or the presence of air currents directly, so these relationships could not be tested.

The expected tidal variation was found to be consistent with measurements of gravity in Oklahoma, USA. Unfortunately the magnitude of the effect is less than was originally thought, so detecting its influence on the clock will be difficult. Spectral analysis was attempted with 30 days of data, but with no result.

Consideration of the theory of spectral analysis, in relation to the amplitudes of the background noise and tidal variation, lead to predictions about how much data will in fact be required to detect the tidal variation: probably at least 45 days. Furthermore, spectral analysis of simulated data suggests that even more ($\sim$100 days) may be required.

There are various events in the recorded data which have not yet been explained. Some of these may be due to air currents, for which there is a plausible mechanism; measurements of air flows are needed to confirm or disprove this suggestion.

7.3 Future work

The basic monitoring system is now in place, but there is plenty of scope for further work. There are several more quantities it would be interesting to measure, which might help to explain some of the unexpected events seen in the data. For example,

- Measuring the air temperature in additional locations, to establish the temperature distribution around the pendulum;
- A contactless infrared temperature sensor could directly measure the temperature of the outermost layer (and possibly second-outermost layer, through the ventilation holes) of the pendulum;
- A means of measuring air flow around the pendulum would allow the effect of air currents to be tested. Whether the doors of the clock enclosure are open could have a large effect on any air currents, so their state could also be measured.
- There are many other options, for example a seismograph, accelerometers, etc.

Audio recording would be a worthwhile addition, for two reasons. It would make a nice addition to the website to be able to hear the clock tick and strike, but more importantly it could allow the monitoring program to check its value of drift. The signal from the sensor only defines the drift modulo 1 second, so listening for the clock striking the quarters or the hour could provide the required integer part of the drift.

There is plenty of room for further analysis. Hopefully once more data is available, the tidal variation will be seen. The effect of gravitational forces on the amplitude of the pendulum has not been analysed, so it is possible that, by bad fortune, variations in gravity could affect the amplitude of the pendulum in just the right way to cancel out the effect on going; this is unlikely however. There are several other events in the recorded data which have not yet been explained.

The website could be developed further, for example to show Fourier transforms of the data or to include a webcam. Finally, automatic alarms to warn the clock keeper of unexpectedly large errors, or if the clock has not been wound, could also be provided and would be appreciated.

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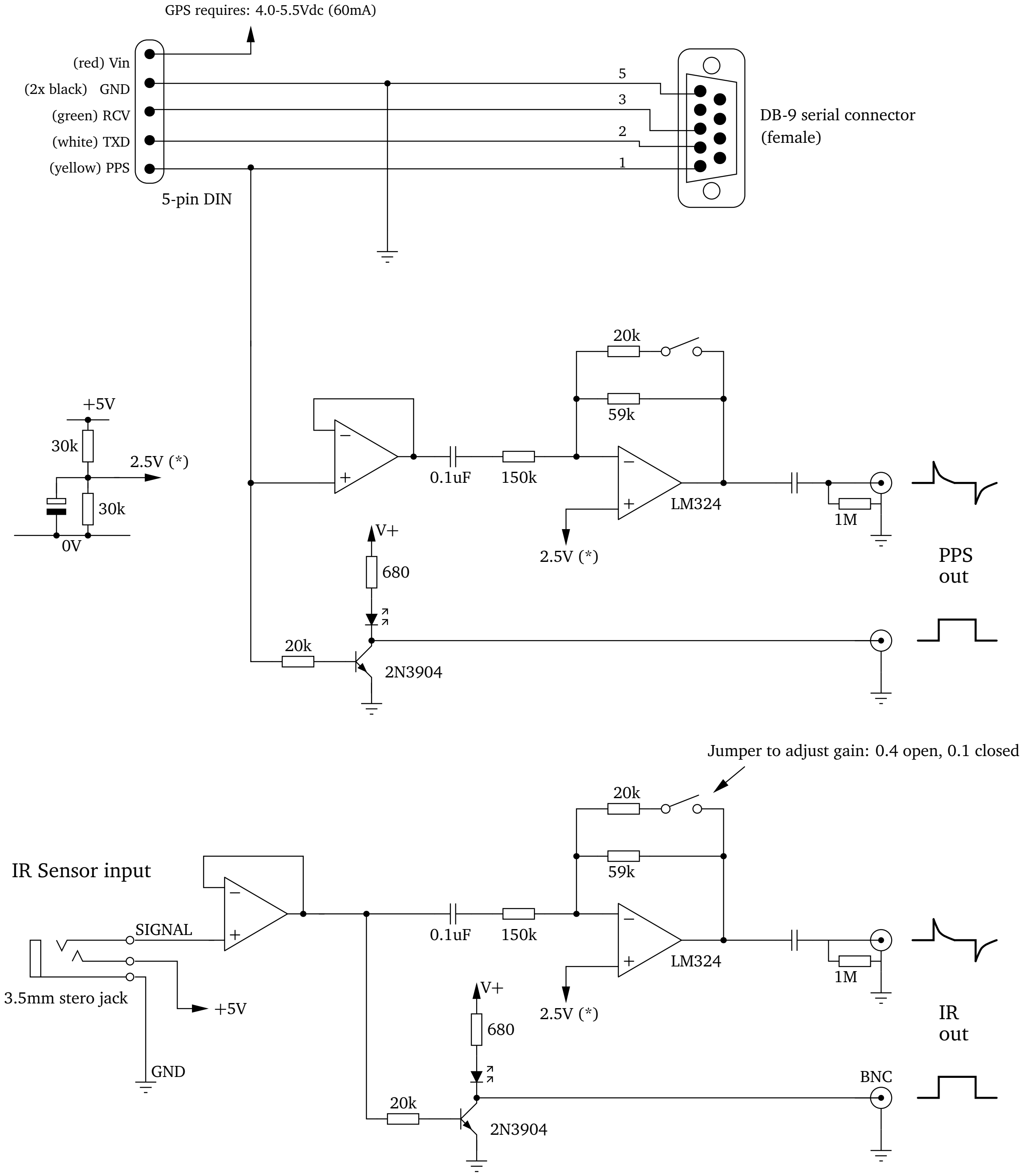
A Interface circuit diagram



Notes:

1. The filtered outputs are intended to be connected to the datalogger. The square wave outputs should be used with a soundcard because it has its own filtering.

2. The PPS signal *rises* to indicate the start of each second, but because of the inverting amplifier this corresponds to a *falling* edge in the filtered output.

FIGURE A.1: Interface circuit, providing power supply to GPS and IR sensor, high-pass filtering and serial interface to GPS.

B Derivation of the period of a non-linear pendulum

The equation of motion of a pendulum is

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0 \tag{B.1}$$

where $\omega_0^2 = g/L$. For small oscillations, $\sin\theta \approx \theta$ and the motion is harmonic with angular frequency ω_0. If the angle of swing is not small, the angular frequency will become ω, but as a first approximation the motion can still be assumed to be sinusoidal, so $\theta \approx A\sin\omega t$. Going back to the equation of motion (B.1), the $\sin\theta$ term may be expanded to give

$$\ddot{\theta} + \omega_0^2\left(\theta - \frac{\theta^3}{6} + \ldots\right) = 0 \tag{B.2}$$

$$-A\omega^2 \sin\omega t + \omega_0^2\left(A\sin\omega t - \frac{A^3 \sin^3\omega t}{6} + \ldots\right) = 0 \tag{B.3}$$

$\sin^3\omega t$ may be expanded as a Fourier series:

$$\sin^3\omega t = a_1 \sin\omega t + a_2 \sin 2\omega t + \ldots$$

where the Fourier coefficients a_n are given by

$$a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\sin(n\theta)\,\mathrm{d}\theta$$

$$\therefore \quad a_1 = \frac{2}{\pi}\int_0^\pi \sin^4\omega t\,\mathrm{d}t = \frac{2}{\pi}\cdot\frac{3\pi}{8} = \frac{3}{4}$$

$$\therefore \quad \sin^3\omega t = \frac{3}{4}\sin\omega t + \ldots \tag{B.4}$$

So, putting (B.4) into (B.3), the approximate equation of motion is

$$-A\omega^2 \sin\omega t + \omega_0^2\left(A\sin\omega t - \frac{A^3}{6}\cdot\frac{3\sin\omega t}{4}\right) \approx 0$$

$$\therefore \quad \omega^2 \approx \omega_0^2\left(1 - \frac{A^2}{8}\right) \tag{B.5}$$

And hence, since $T = 2\pi/\omega$,

$$T \approx T_0\left(1 + \frac{A^2}{16}\right) \tag{B.6}$$

C Temperature compensation

C.1 Response to ramp input

Using the model of a temperature compensated pendulum shown in Figure 4.4, we can derive the response of the pendulum to a transient change in temperature, for example a ramp. Assume that this change in ambient temperature affects the temperature of the outer layer T_1 directly, so the input is

$$T_1 = T_0 + \frac{dT_1}{dt} t$$

If the layers are linked by a thermal resistance R, and have heat capacity mc_p, the heat flow between them is

$$Q = \frac{T_1 - T_2}{R}$$

and the heat flow into the inner layer is

$$Q = mc_p \frac{dT_2}{dt}$$

Equating these gives the governing equation:

$$\tau \frac{dT_2}{dt} + T_2 = T_1 \qquad \text{where } \tau = mc_p R$$

which can be solved (e.g. by Laplace transforms) to give the response to a ramp in T_1,

$$T_2 = \frac{dT_1}{dt} \left[(t - \tau) + \tau e^{-t/\tau} \right] + T_0$$

Thus the temperature difference, once steady state is reached, is

$$T_1 - T_2 = \tau \frac{dT_1}{dt}$$

C.2 Estimation of constants

The steel (density 7800 kg/m^3, heat capacity 460 J/kgK, expansion coefficient 13×10^{-6} K^{-1}) outer layer of the pendulum is approximately 35mm diameter, 2.2m long and 3mm thick. Its mass is

$$m = \rho \pi d l t = 5.7 \text{ kg}$$

The thermal resistance R consists of the surface convection resistances (very variable, but say[1] about 0.2 mK/W) and the air itself (conductivity[2] 0.0257 W/mK). The air gap is approximately 1mm, giving

$$\text{Resistivity } \frac{1}{U} = 2 \times 0.2 + \frac{1 \times 10^{-3}}{0.0257} = 0.44 \text{ m}^2\text{K/W}$$

[1] from 4D11 Building Physics notes

[2] from http://www.engineeringtoolbox.com/

$$\implies \quad \text{Resistance } R = \frac{1}{UA} = \frac{0.44}{\pi \times 28 \times 10^{-3} \times 2.2} = 2.3 \text{ K/W}$$

So the time constant is

$$\tau = mc_p R = 5.7 \times 460 \times 2.0 = 6030 \text{ seconds} \approx 1.5 \text{ hours}$$

The steady temperature difference is given by

$$T_1 - T_2 = \tau \frac{dT_1}{dt}$$

so the change in going caused by this is

$$\begin{aligned}
\Delta G &= \frac{-\Delta T}{T} = \frac{-1}{2} \cdot \frac{\Delta L}{L} = -\frac{\alpha}{2}(T_1 - T_2) \\
&= \frac{-\alpha\tau}{2}\frac{dT_1}{dt} \\
&= -k\frac{dT_1}{dt}
\end{aligned}$$

$$\implies \quad k = \frac{1}{2} \cdot 13 \times 10^{-6} \cdot 6030 \approx 40 \text{ ms/degree}$$

D Derivation of gravitational effects

Consider first the effect of the Moon alone; the derivation below applies equally to the effect of the Sun. The are two relevant forces which act on an object on the Earth: the gravitational pull of the Moon, and the D'Alembert force corresponding to the centripetal acceleration of the Earth's orbit with the Moon.

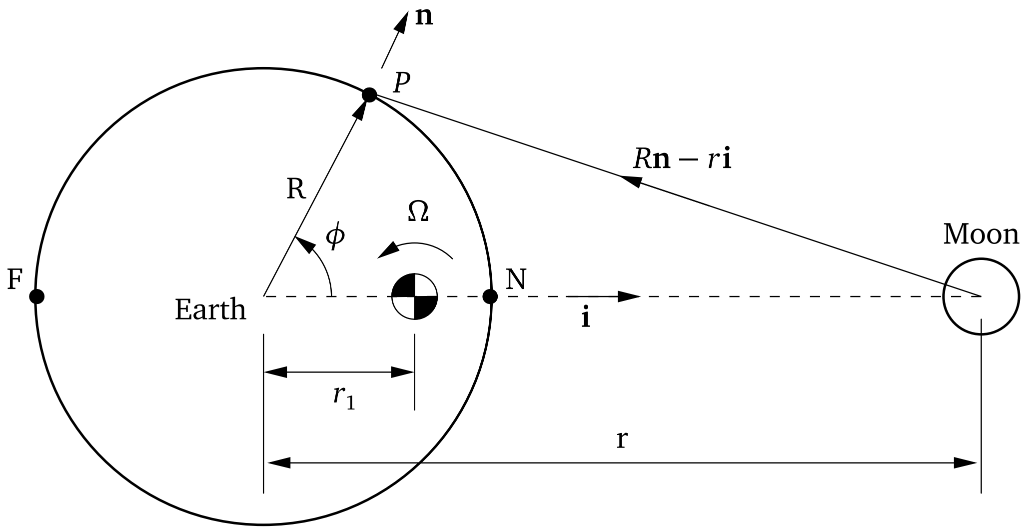


FIGURE D.1: Plan view of Earth and Moon: points F, N and P lie on the equator.

Gravitational pull Using Newton's Law of Gravitation, $F = GMm/r^2$, the force felt by a point on the Earth varies with the inverse square of distance. As a vector, the force felt by an object of unit mass at point P is

$$\mathbf{F} = \frac{GM}{|R\mathbf{n} - r\mathbf{i}|^2} \cdot \frac{-(R\mathbf{n} - r\mathbf{i})}{|R\mathbf{n} - r\mathbf{i}|}$$

and using the cosine rule,

$$|R\mathbf{n} - r\mathbf{i}|^2 = R^2 + r^2 - 2rR\cos\phi$$

so

$$\mathbf{F} = \frac{-GM(R\mathbf{n} - r\mathbf{i})}{\left[R^2 + r^2 - 2rR\cos\phi\right]^{3/2}}$$

If $R \ll r$, then we can use the binomial expansion to give

$$\begin{aligned}\left[R^2 + r^2 - 2rR\cos\phi\right]^{-3/2} &\approx r^{-3}\left[1 - \frac{3}{2}\left(\frac{R^2}{r^2} - 2\frac{R}{r}\cos\phi\right)\right] \\ &= r^{-3}\left[1 + 3\frac{R}{r}\cos\phi\right] + \mathrm{O}\left\{(R/r)^2\right\}\end{aligned}$$

We are interested in the *downwards* force, which is given by

$$\begin{aligned}\mathbf{F}\cdot(-\mathbf{n}) &= \frac{GM}{r^3}\left[1 + 3\frac{R}{r}\cos\phi\right](R - r\cos\phi) + \mathrm{O}\left\{(R/r)^2\right\} \\ &= \frac{GM}{r^2}\left[\frac{R}{r} - \cos\phi - 3\frac{R}{r}\cos^2\phi\right] + \mathrm{O}\left\{(R/r)^2\right\} \\ &\approx \frac{-GM}{r^2}\left[\frac{1}{2}\frac{R}{r} + \cos\phi + \frac{3}{2}\frac{R}{r}\cos 2\phi\right]\end{aligned}$$

(since $\mathbf{n}\cdot\mathbf{i} = \cos\phi$).

Centrifugal force Since the Earth and Moon are orbiting each other, about a centre of rotation (the "barycentre") located somewhere between them, the gravitational force between them must balance the centrifugal force of their rotation, i.e.

$$F = \frac{GM_eM_m}{r^2} = M_e r_1 \Omega^2 = M_m r_2 \Omega^2$$

$$\implies \quad r_1\Omega^2 = \frac{GM_m}{r^2}$$

Now consider the point P, whose position relative to the barycentre is

$$\begin{aligned}\mathbf{r_P} &= R\mathbf{n} - r_1\mathbf{i} \\ \implies \quad \ddot{\mathbf{r}}_\mathbf{P} &= -\Omega^2 R\mathbf{n} + \Omega^2 r_1\mathbf{i} = -\Omega^2\mathbf{r_P} \\ \therefore \quad \text{D'Alembert force } \mathbf{F} &= m\Omega^2\mathbf{r_P}\end{aligned}$$

So the downwards force experienced by an object of unit mass at P is

$$\begin{aligned}\Omega^2 \mathbf{r_P} \cdot (-\mathbf{n}) &= -\Omega^2 \left[R - r_1 \cos\phi\right] \\ &= r_1 \Omega^2 \left[\cos\phi - \frac{R}{r_1}\right] \\ &= \frac{GM_m}{r^2}\left[\cos\phi - \frac{R}{r_1}\right]\end{aligned}$$

Total force So, the total downwards force per unit mass is

$$\begin{aligned}F_{down} &= \frac{-GM_m}{r^2}\left[\frac{1}{2}\frac{R}{r} + \cos\phi + \frac{3}{2}\frac{R}{r}\cos 2\phi\right] + \frac{GM_m}{r^2}\left[\cos\phi - \frac{R}{r_1}\right] \\ &= \frac{-GM_m}{r^2}\left[\frac{1}{2}\frac{R}{r} + \frac{3}{2}\frac{R}{r}\cos 2\phi + \frac{R}{r_1}\right] \\ &= \frac{-1}{2}\cdot\frac{GM_m}{r^2}\frac{R}{r}\left[1 + 3\cos 2\phi + \frac{r}{r_1}\right]\end{aligned}$$

A downwards force per unit mass is effectively a change in gravity, so taking $\Delta g = F_{down}$ and $g = (GM_e)/R^2$,

$$\begin{aligned}\frac{\Delta g}{g} &= \frac{R^2}{GM_e} \times \frac{-GM_m R}{r^3}\left[1 + 3\cos 2\phi + \frac{r}{r_1}\right] \\ &= \frac{M_m}{M_e}\left(\frac{R}{r}\right)^3\left[1 + 3\cos 2\phi + \frac{r}{r_1}\right]\end{aligned}$$

Finally, the barycentre (which is the centre of mass of the Earth-Moon system) can be located by considering 'moments of mass' around the Earth:

$$r_1(M_m + M_e) = r M_m \quad \Longrightarrow \quad \frac{r}{r_1} = \frac{M_m + M_e}{M_m} = \frac{\lambda + 1}{\lambda}$$

where $\lambda = M_m/M_e$. So

$$\begin{aligned}\frac{\Delta g}{g} &= \lambda\left(\frac{R}{r}\right)^3\left[1 + 3\cos 2\phi + \frac{\lambda+1}{\lambda}\right] \\ &= \left(\frac{R}{r}\right)^3\left[1 + 2\lambda + 3\lambda\cos 2\phi\right]\end{aligned}$$

The change in gravity due to the Moon is superimposed on that due to the Sun, to give the beating effect shown in Figure 4.5.

E Risk assessment retrospective

This project is almost entirely computer-based, and no specific hazards were identified apart from the usual hazards of such work. These were addressed by ensuring computer working areas were arranged comfortably, and in retrospect this seems to have been an appropriate assessment.