At a typical pressure of 1010 hPa, this gives $dA/dp = -0.018\,\text{mrad/hPa}$ and $dG/dp = 0.12\,\text{ms}\,\text{day}^{-1}/\text{hPa}$.

5 Extracting tidal variations in gravity

We saw in Section 4.6.1 that the tidal variation in gravity will cause a periodic change in going of at most ±10 milliseconds per day. This is tiny compared to the noise in the measurements of going, which has an RMS value of about 600 ms/day. Nonetheless spectral analysis, given a sufficiently long period of data, should be able to discern this periodic variation from the noise. But how long is 'sufficiently long'?

The first condition is related to the duration of the data. With a sample rate F_s and data of duration T, the number of samples is $N = F_s T$. The number of points in the spectrum is also N, so the bin width $\Delta f = F_s/N = 1/T$. The tidal variation occurs at a frequency of approximately 2 cycles per day, the frequency bins must be sufficiently narrow for this to be distinguishable from lower frequency variations. For example, if we want to find the tidal variation in the 10th bin, we need $\Delta f = 0.2$ cycles per day. This means T must be at least 5 days.

The second condition is that the tidal variation be visible above the noise. 'White' noise has a flat spectrum (Figure 5.1), and it is a property of spectra that the area under the spectrum is equal to the mean square (MS) of the time-domain signal. If the maximum frequency content of the noise is F_N and the amplitude of the spectrum is B^2 then

$$\text{Area under spectrum} = 2 \times 2\pi F_N \times B^2 = \text{MS}_{\text{noise}}$$
$$\implies \quad B^2 = \frac{\text{MS}_{\text{noise}}}{4\pi F_N}$$

The Fourier transform of the periodic tidal variation (with RMS value A) is $\frac{A^2}{2}\delta(\omega+\omega_0) + \frac{A^2}{2}\delta(\omega-\omega_0)$. In a real spectrum, we cannot have infinitely thin δ-functions, so the best approximation will be that the spike falls into only one frequency bin of width $\Delta\omega$ and height C^2, say. The areas of the bin and the δ-function must be equal, i.e.

$$C^2\Delta\omega = \frac{A^2}{2}\int_{-\infty}^{+\infty}\delta(\omega-\omega_0)\,\mathrm{d}\omega = \frac{A^2}{2}$$

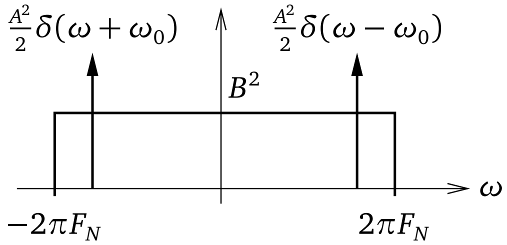

FIGURE 5.1: Spectrum of $A\sqrt{2}\cos\omega t$ and white noise, RMS $= B^2$

$$\implies \quad C^2 = \frac{A^2}{2\Delta\omega} = \frac{A^2}{4\pi} \cdot \frac{1}{\Delta f} = \frac{A^2 T}{4\pi}$$

In practical spectral analysis, a window is used to reduce the effect of having a finite-length non-periodic signal. This reduces spectral leakage, at the expense of reducing the peak height of a spike. For example, the Hann window reduces the peak value by one half. So be able to discern the tidal variation in the spectrum above the noise, and including a factor of one half to compensate for the window, we need

$$\begin{aligned} C^2 &> 2 \times B^2 \\ \frac{A^2 T}{4\pi} &> \frac{2 \times \mathrm{MS}_{\text{noise}}}{4\pi F_N} \\ T &> \frac{2 \times \mathrm{MS}_{\text{noise}}}{A^2 F_N} \end{aligned}$$

or, putting $T = N/F_s$,

$$N > \frac{2 \times \mathrm{MS}_{\text{noise}}}{A^2} \cdot \frac{F_s}{F_N}$$

If the noise has a wide frequency range, we could assume it has frequency content right up to the Nyquist frequency, i.e. $F_N = F_{nyq} = F_s/2$. Then

$$N > \frac{4 \times \mathrm{MS}_{\text{noise}}}{A^2}$$

Putting values into this, the maximum RMS value of the tidal variation is $A = \frac{10}{\sqrt{2}} =$ 7 ms/day, and that of the noise is 600 ms/day. So we need

$$N > \frac{4 \times 600^2}{7^2} = 3 \times 10^4$$

which, at a sample rate of once every 3 seconds, is about 1 day. When the tidal variation is at its minimum of about ±4 ms/day, the requirement is increased to $N > 9 \times 10^4$, or 3 days.

Next is the question of accuracy. Even if the noise was roughly 'white,' its spectrum will itself be noisy because we are sampling it over a finite period. Newland [8, Ch.9] shows that the expected accuracy of a real spectrum is

$$\frac{\sigma}{m} \approx \frac{1}{\sqrt{B_e T}}$$

where m and σ are the mean and standard deviation of the spectrum, B_e is the effective bandwidth of the spectral window and T is the record length, as above. The effective bandwidth depends on the shape of the window which is used, but it is given roughly by $B_e \approx 1/T$. T can be increased without increasing B_e, to get better accuracy, by averaging. So, to obtain an accuracy of say $\sigma/m = 0.1$, we need $B_e T = 100$, i.e. 100-fold averaging.

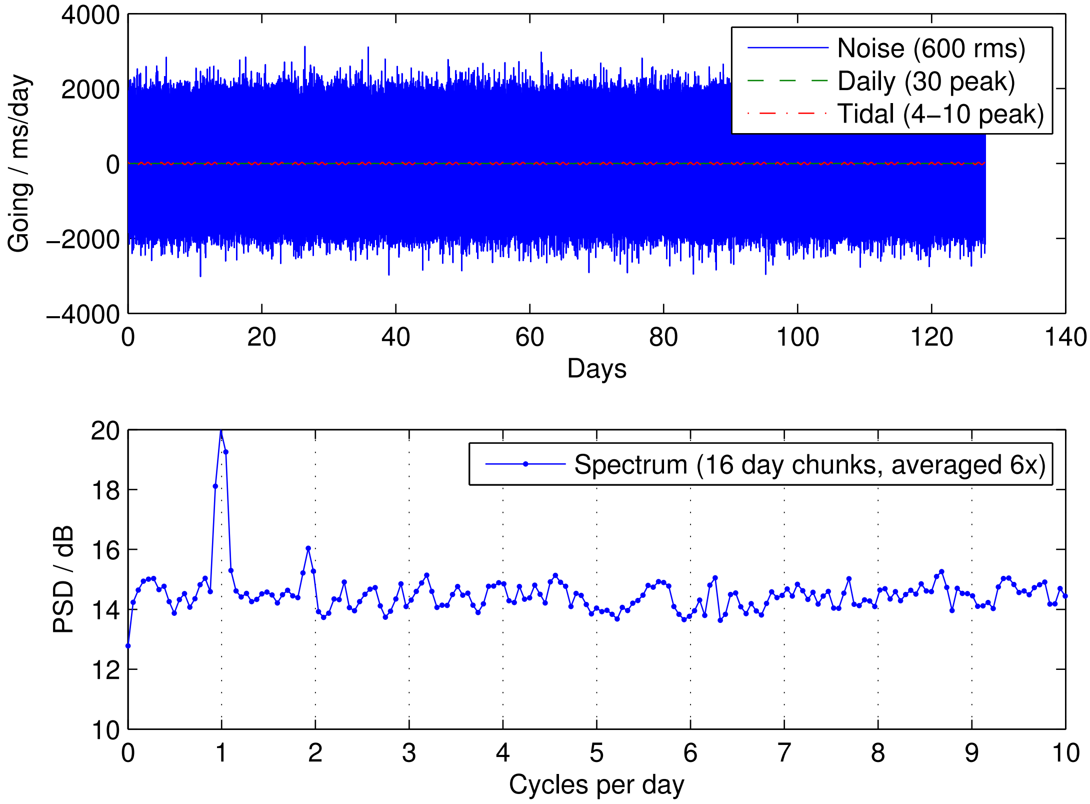

FIGURE 5.2: Spectral analysis of simulated tidal variation.

With $B_e = 0.2$ cycles per day as above, this means we require $T = 500$ days. This is clearly quite a long time.

To get satisfactory results with less data, we could either require less accuracy or reduce the level of the noise. An accuracy of 10% is probably more than sufficient to detect the tidal variation, so we could settle for 33%, giving $B_e T = 9$ and $T = 45$ days — this is much more manageable.

The noise level can be reduced by calculating the going over a longer interval, but at the expense of a reduced sample rate. For example, calculating the going every 30 seconds instead of 3 seconds gives an RMS noise of about 280 ms / day. Thus the number of points needed is

$$N > \frac{4 \times 280^2}{7^2} = 6400$$

which is 2.2 days. So increasing the interval between data points requires a *longer* period of data, even though the noise level is decreased.

5.1 Simulation

Simulated noise and tidal signals were used to do some virtual experimentation to check the above results. The RMS value of the noise in going was estimated as 600 ms / day, and a tidal effect of between 4 and 10 ms / day peak was added in, along with a generic daily variation. There are many combinations of possible windows, data lengths and averaging, but the one shown in Figure 5.2 seems to give good results from minimal data, while still reliably and clearly identifying the tidal signal: this simulation used 96 days of data in 16 day chunks, averaged 6 times, using a Hamming window. This suggests that the calculations above may underestimate the amount of data required to reliably identify the tidal variation.