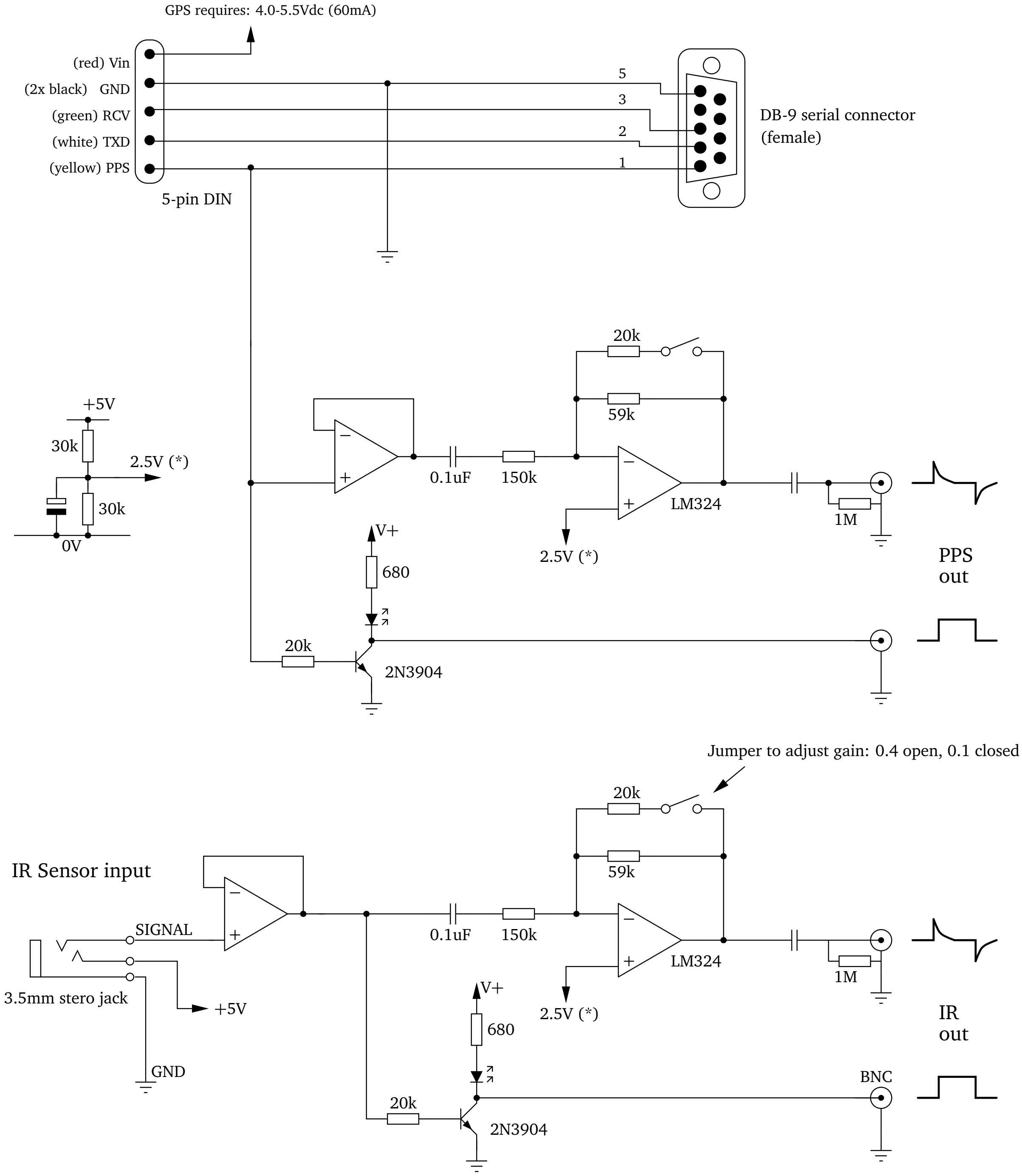
A Interface circuit diagram



Notes:

1. The filtered outputs are intended to be connected to the datalogger. The square wave outputs should be used with a soundcard because it has its own filtering.

2. The PPS signal *rises* to indicate the start of each second, but because of the inverting amplifier this corresponds to a *falling* edge in the filtered output.

FIGURE A.1: Interface circuit, providing power supply to GPS and IR sensor, high-pass filtering and serial interface to GPS.

B Derivation of the period of a non-linear pendulum

The equation of motion of a pendulum is

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0 \tag{B.1}$$

where $\omega_0^2 = g/L$. For small oscillations, $\sin\theta \approx \theta$ and the motion is harmonic with angular frequency ω_0. If the angle of swing is not small, the angular frequency will become ω, but as a first approximation the motion can still be assumed to be sinusoidal, so $\theta \approx A\sin\omega t$. Going back to the equation of motion (B.1), the $\sin\theta$ term may be expanded to give

$$\begin{aligned} \ddot{\theta} + \omega_0^2\left(\theta - \frac{\theta^3}{6} + \ldots\right) &= 0 && \text{(B.2)} \\ -A\omega^2 \sin\omega t + \omega_0^2\left(A\sin\omega t - \frac{A^3\sin^3\omega t}{6} + \ldots\right) &= 0 && \text{(B.3)} \end{aligned}$$

$\sin^3\omega t$ may be expanded as a Fourier series:

$$\sin^3\omega t = a_1 \sin\omega t + a_2 \sin 2\omega t + \ldots$$

where the Fourier coefficients a_n are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin(n\theta)\,\mathrm{d}\theta$$

$$\therefore \quad a_1 = \frac{2}{\pi}\int_0^{\pi} \sin^4\omega t\,\mathrm{d}t = \frac{2}{\pi}\cdot\frac{3\pi}{8} = \frac{3}{4}$$

$$\therefore \quad \sin^3\omega t = \frac{3}{4}\sin\omega t + \ldots \tag{B.4}$$

So, putting (B.4) into (B.3), the approximate equation of motion is

$$-A\omega^2 \sin\omega t + \omega_0^2\left(A\sin\omega t - \frac{A^3}{6}\cdot\frac{3\sin\omega t}{4}\right) \approx 0$$

$$\therefore \quad \omega^2 \approx \omega_0^2\left(1 - \frac{A^2}{8}\right) \tag{B.5}$$

And hence, since $T = 2\pi/\omega$,

$$T \approx T_0\left(1 + \frac{A^2}{16}\right) \tag{B.6}$$

C Temperature compensation

C.1 Response to ramp input

Using the model of a temperature compensated pendulum shown in Figure 4.4, we can derive the response of the pendulum to a transient change in temperature, for example a ramp. Assume that this change in ambient temperature affects the temperature of the outer layer T_1 directly, so the input is

$$T_1 = T_0 + \frac{dT_1}{dt} t$$

If the layers are linked by a thermal resistance R, and have heat capacity mc_p, the heat flow between them is

$$Q = \frac{T_1 - T_2}{R}$$

and the heat flow into the inner layer is

$$Q = mc_p \frac{dT_2}{dt}$$

Equating these gives the governing equation:

$$\tau \frac{dT_2}{dt} + T_2 = T_1 \qquad \text{where } \tau = mc_p R$$

which can be solved (e.g. by Laplace transforms) to give the response to a ramp in T_1,

$$T_2 = \frac{dT_1}{dt} \left[(t - \tau) + \tau e^{-t/\tau} \right] + T_0$$

Thus the temperature difference, once steady state is reached, is

$$T_1 - T_2 = \tau \frac{dT_1}{dt}$$

C.2 Estimation of constants

The steel (density 7800 kg/m^3, heat capacity 460 J/kgK, expansion coefficient 13×10^{-6} K^{-1}) outer layer of the pendulum is approximately 35mm diameter, 2.2m long and 3mm thick. Its mass is

$$m = \rho \pi d l t = 5.7 \text{ kg}$$

The thermal resistance R consists of the surface convection resistances (very variable, but say[1] about 0.2 mK/W) and the air itself (conductivity[2] 0.0257 W/mK). The air gap is approximately 1mm, giving

$$\text{Resistivity } \frac{1}{U} = 2 \times 0.2 + \frac{1 \times 10^{-3}}{0.0257} = 0.44 \text{ m}^2\text{K/W}$$

[1] from 4D11 Building Physics notes

[2] from `http://www.engineeringtoolbox.com/`

$$\implies \quad \text{Resistance } R = \frac{1}{UA} = \frac{0.44}{\pi \times 28 \times 10^{-3} \times 2.2} = 2.3 \text{ K/W}$$

So the time constant is

$$\tau = mc_pR = 5.7 \times 460 \times 2.0 = 6030 \text{ seconds} \approx 1.5 \text{ hours}$$

The steady temperature difference is given by

$$T_1 - T_2 = \tau \frac{dT_1}{dt}$$

so the change in going caused by this is

$$\begin{aligned} \Delta G &= \frac{-\Delta T}{T} = \frac{-1}{2} \cdot \frac{\Delta L}{L} = -\frac{\alpha}{2}(T_1 - T_2) \\ &= \frac{-\alpha\tau}{2}\frac{dT_1}{dt} \\ &= -k\frac{dT_1}{dt} \end{aligned}$$

$$\implies \quad k = \frac{1}{2} \cdot 13 \times 10^{-6} \cdot 6030 \approx 40 \text{ ms/degree}$$

D Derivation of gravitational effects

Consider first the effect of the Moon alone; the derivation below applies equally to the effect of the Sun. The are two relevant forces which act on an object on the Earth: the gravitational pull of the Moon, and the D'Alembert force corresponding to the centripetal acceleration of the Earth's orbit with the Moon.

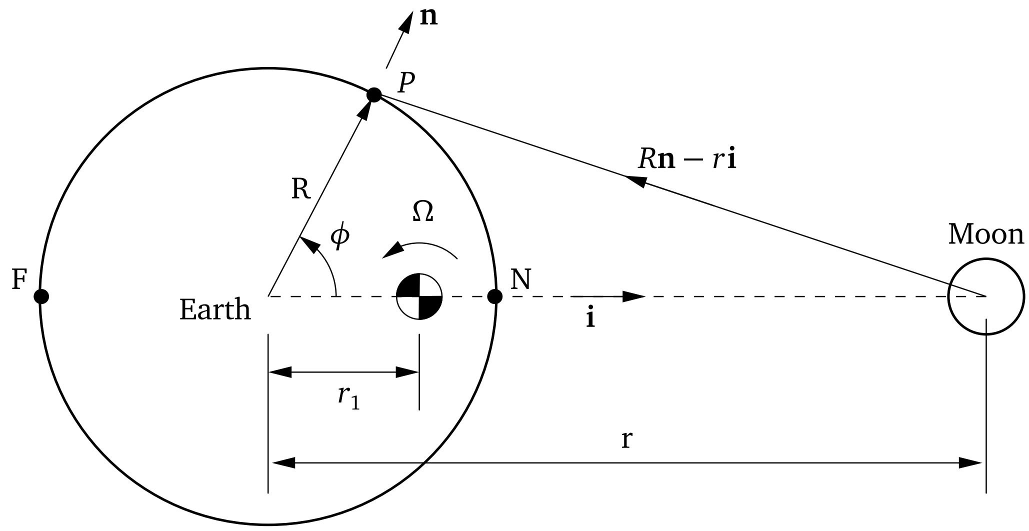


FIGURE D.1: Plan view of Earth and Moon: points F, N and P lie on the equator.

Gravitational pull Using Newton's Law of Gravitation, $F = GMm/r^2$, the force felt by a point on the Earth varies with the inverse square of distance. As a vector, the force felt by an object of unit mass at point P is

$$\mathbf{F} = \frac{GM}{|R\mathbf{n} - r\mathbf{i}|^2} \cdot \frac{-(R\mathbf{n} - r\mathbf{i})}{|R\mathbf{n} - r\mathbf{i}|}$$

and using the cosine rule,

$$|R\mathbf{n} - r\mathbf{i}|^2 = R^2 + r^2 - 2rR\cos\phi$$

so

$$\mathbf{F} = \frac{-GM(R\mathbf{n} - r\mathbf{i})}{\left[R^2 + r^2 - 2rR\cos\phi\right]^{3/2}}$$

If $R \ll r$, then we can use the binomial expansion to give

$$\begin{aligned} \left[R^2 + r^2 - 2rR\cos\phi\right]^{-3/2} &\approx r^{-3}\left[1 - \frac{3}{2}\left(\frac{R^2}{r^2} - 2\frac{R}{r}\cos\phi\right)\right] \\ &= r^{-3}\left[1 + 3\frac{R}{r}\cos\phi\right] + \mathrm{O}\left\{({}^R\!/_r)^2\right\} \end{aligned}$$

We are interested in the *downwards* force, which is given by

$$\begin{aligned} \mathbf{F}\cdot(-\mathbf{n}) &= \frac{GM}{r^3}\left[1 + 3\frac{R}{r}\cos\phi\right](R - r\cos\phi) + \mathrm{O}\left\{({}^R\!/_r)^2\right\} \\ &= \frac{GM}{r^2}\left[\frac{R}{r} - \cos\phi - 3\frac{R}{r}\cos^2\phi\right] + \mathrm{O}\left\{({}^R\!/_r)^2\right\} \\ &\approx \frac{-GM}{r^2}\left[\frac{1}{2}\frac{R}{r} + \cos\phi + \frac{3}{2}\frac{R}{r}\cos 2\phi\right] \end{aligned}$$

(since $\mathbf{n}\cdot\mathbf{i} = \cos\phi$).

Centrifugal force Since the Earth and Moon are orbiting each other, about a centre of rotation (the "barycentre") located somewhere between them, the gravitational force between them must balance the centrifugal force of their rotation, i.e.

$$F = \frac{GM_eM_m}{r^2} = M_e r_1 \Omega^2 = M_m r_2 \Omega^2$$

$$\implies \quad r_1\Omega^2 = \frac{GM_m}{r^2}$$

Now consider the point P, whose position relative to the barycentre is

$$\begin{aligned} \mathbf{r_P} &= R\mathbf{n} - r_1\mathbf{i} \\ \implies \quad \ddot{\mathbf{r}}_\mathbf{P} &= -\Omega^2 R\mathbf{n} + \Omega^2 r_1\mathbf{i} = -\Omega^2\mathbf{r_P} \\ \therefore \quad \text{D'Alembert force } \mathbf{F} &= m\Omega^2\mathbf{r_P} \end{aligned}$$

So the downwards force experienced by an object of unit mass at P is

$$\begin{aligned}\Omega^2 \mathbf{r_P} \cdot (-\mathbf{n}) &= -\Omega^2 \left[R - r_1 \cos\phi\right] \\ &= r_1 \Omega^2 \left[\cos\phi - \frac{R}{r_1}\right] \\ &= \frac{GM_m}{r^2}\left[\cos\phi - \frac{R}{r_1}\right]\end{aligned}$$

Total force So, the total downwards force per unit mass is

$$\begin{aligned}F_{down} &= \frac{-GM_m}{r^2}\left[\frac{1}{2}\frac{R}{r} + \cos\phi + \frac{3}{2}\frac{R}{r}\cos 2\phi\right] + \frac{GM_m}{r^2}\left[\cos\phi - \frac{R}{r_1}\right] \\ &= \frac{-GM_m}{r^2}\left[\frac{1}{2}\frac{R}{r} + \frac{3}{2}\frac{R}{r}\cos 2\phi + \frac{R}{r_1}\right] \\ &= \frac{-1}{2} \cdot \frac{GM_m}{r^2}\frac{R}{r}\left[1 + 3\cos 2\phi + \frac{r}{r_1}\right]\end{aligned}$$

A downwards force per unit mass is effectively a change in gravity, so taking $\Delta g = F_{down}$ and $g = (GM_e)/R^2$,

$$\begin{aligned}\frac{\Delta g}{g} &= \frac{R^2}{GM_e} \times \frac{-GM_m R}{r^3}\left[1 + 3\cos 2\phi + \frac{r}{r_1}\right] \\ &= \frac{M_m}{M_e}\left(\frac{R}{r}\right)^3\left[1 + 3\cos 2\phi + \frac{r}{r_1}\right]\end{aligned}$$

Finally, the barycentre (which is the centre of mass of the Earth-Moon system) can be located by considering 'moments of mass' around the Earth:

$$r_1(M_m + M_e) = r M_m \qquad \Longrightarrow \qquad \frac{r}{r_1} = \frac{M_m + M_e}{M_m} = \frac{\lambda + 1}{\lambda}$$

where $\lambda = M_m/M_e$. So

$$\begin{aligned}\frac{\Delta g}{g} &= \lambda\left(\frac{R}{r}\right)^3\left[1 + 3\cos 2\phi + \frac{\lambda+1}{\lambda}\right] \\ &= \left(\frac{R}{r}\right)^3\left[1 + 2\lambda + 3\lambda\cos 2\phi\right]\end{aligned}$$

The change in gravity due to the Moon is superimposed on that due to the Sun, to give the beating effect shown in Figure 4.5.

E Risk assessment retrospective

This project is almost entirely computer-based, and no specific hazards were identified apart from the usual hazards of such work. These were addressed by ensuring computer working areas were arranged comfortably, and in retrospect this seems to have been an appropriate assessment.